Some Fundamental Concepts of Topology in Terms of the Algebra of Logic

by

Miron Zarycki (Lviv)

Author's note. I would like to thank Dr. Kuratowski for his invaluable advice while writing the final version of this article.

Note. This English translation of Zarycki's paper *Quelques notions fondamentales de l'Analysis Situs au point de vue de l'Algèbre de la Logique* (in French), Fund. Math. **9**(1927), 3–15, was prepared by Mark Bowron in July 2012.

The first part of Kuratowski's doctoral dissertation **[2]** is devoted to an analysis of the notion of *closure* of a set. The *topological closure* of A, which we denote by A^r, is the union of A and the set of all limit points of A.

The following four *topological closure axioms* form the basis of all of Kuratowski's arguments:

$$\begin{aligned} I_r\colon&\quad (A \cup B)^r = A^r \cup B^r \\ II_r\colon&\quad A \subset A^r \\ III_r\colon&\quad \varnothing^r = \varnothing \\ IV_r\colon&\quad A^{rr} = A^r \end{aligned}$$

In this article we present similar systems of axioms for some other basic topological concepts, namely the *exterior*, the *interior*, the *frontier*, and the *border*. We establish the equivalence of these systems with that of Kuratowski and deduce some fundamental properties.

§ 1. The Axiom Systems

1. We denote the space by the symbol C and the complement of a set A by A^c. We assume as given four well-defined functions on the power set of C into itself, each of which satisfies a certain system of axioms (see below). These functions shall be denoted by A^e (exterior), A^i (interior), A^f (frontier), and A^b (border).

Here are the systems of axioms denoted, respectively, by S_e, S_i, S_f, and S_b (the closure axioms shall be denoted by S_r):

$$\begin{aligned} I_e\colon&\quad (A \cup B)^e = A^e \cap B^e \\ II_e\colon&\quad A^e \subset A^c \\ III_e\colon&\quad \varnothing^e = C \\ IV_e\colon&\quad A^{ece} = A^e \end{aligned}$$

$$\begin{aligned} I_i\colon&\quad (A \cap B)^i = A^i \cap B^i \\ II_i\colon&\quad A^i \subset A \\ III_i\colon&\quad C^i = C \\ IV_i\colon&\quad A^{ii} = A^i \end{aligned}$$

$$\begin{aligned} I_f\colon&\quad (A \cap B) \cap (A \cap B)^f = (A \cap B) \cap (A^f \cup B^f) \\ II_f\colon&\quad A^f = A^{cf} \\ III_f\colon&\quad \varnothing^f = \varnothing \\ IV_f\colon&\quad A^{ff} \subset A^f \end{aligned}$$

$$\begin{aligned} I_b\colon&\quad (A \cap B)^b = (A \cap B^b) \cup (B \cap A^b) \\ II_b\colon&\quad C^b = \varnothing \\ III_b\colon&\quad A^{cbcb} \subset A. \end{aligned}$$

2. Assume $A^e = A^{rc}$, $A^i = A^{crc}$, $A^f = A^r \cap A^{cr}$, and $A^b = A \cap A^{cr}$. Using only these assumptions, the axiom system S_r and the algebra of logic, the following identities are easily verified:

$$A^r = A^{ec} = A^{cic} = A \cup A^f = A \cup A^{cb}$$

$$A^e = A^{rc} = A^{ci} = A^c \cap A^{fc} = A^c \cap A^{cbc}$$

$$A^i = A^{crc} = A^{ce} = A \cap A^{fc} = A \cap A^{bc}$$

$$A^f = A^r \cap A^{cr} = A^{ec} \cap A^{cec} = A^{ic} \cap A^{cic} = A^b \cup A^{cb}$$

$$A^b = A \cap A^{cr} = A \cap A^{cec} = A \cap A^{ic} = A \cap A^f.$$

3. To establish the equivalence of systems S_r, S_e, S_i, S_f and S_b it suffices to prove the following five implications:

$$\begin{aligned} 1^0.\quad & S_r \Longrightarrow S_e \\ 2^0.\quad & S_e \Longrightarrow S_i \\ 3^0.\quad & S_i \Longrightarrow S_f \\ 4^0.\quad & S_f \Longrightarrow S_b \\ 5^0.\quad & S_b \Longrightarrow S_r. \end{aligned}$$

The following five identities [from the previous page] are useful for these proofs: $A^e = A^{rc}$, $A^i = A^{ce}$, $A^f = A^{ic} \cap A^{cic}$, $A^b = A \cap A^f$, and $A^r = A \cup A^{cb}$.

Note. We omit the proofs that do not present any difficulty; we limit ourselves to mentioning the following proofs that are perhaps not immediate:

1) S_i *implies* I_f.

Proof: We have $M^{ci} \subset M^c$ (II_i), thus $M \subset M^{cic}$ (α).

We then obtain:

$$\begin{aligned} (A \cap B)^{ic} &= A^{ic} \cup B^{ic} \text{ (by } I_i\text{), and} \\ (A \cap B) \cap (A \cap B)^{ic} &= (A \cap B \cap A^{ic}) \cup (A \cap B \cap B^{ic}). \end{aligned}$$

Taking into account formula (α) we can write the last equation as follows:

$$\begin{aligned} (A \cap B) \cap (A \cap B)^{cic} \cap (A \cap B)^{ic} &= (A \cap B \cap A^{cic} \cap A^{ic}) \cup (A \cap B \cap B^{cic} \cap B^{ic}) \\ (A \cap B) \cap (A \cap B)^f &= (A \cap B) \cap (A^f \cup B^f). \text{ QED} \end{aligned}$$

2) S_i *implies* IV_f.

Proof: We have $(M \cap N)^i = M^i \cap N^i$ (by I_i). Set $M = A^i$ and $N = A^i \cup B^i$.

We obtain $A^{ii} = A^{ii} \cap (A^i \cup B^i)^i$, or $A^{ii} = A^i \subset (A^i \cup B^i)^i$.

We obtain similarly $B^i \subset (A^i \cup B^i)^i$, therefore $A^i \cup B^i \subset (A^i \cup B^i)^i$.

On the other hand we obtain $(A^i \cup B^i)^i \subset A^i \cup B^i$ (by II_i), or $(A^i \cup B^i)^i = A^i \cup B^i$.

Taking this formula into account, we conclude that:

$A^{ff} = (A^{ic} \cap A^{cic})^{ic} \cap (A^{ic} \cap A^{cic})^{cic} \subset (A^{ic} \cap A^{cic})^{cic} = (A^i \cup A^{ci})^{ic} = (A^i \cup A^{ci})^c = A^{ic} \cap A^{cic} = A^f$. QED

3) S_f *implies* I_b.

Proof:

$(A \cap B)^b = A \cap B \cap (A \cap B)^f = A \cap B \cap (A^f \cup B^f) = (A \cap B \cap B^f) \cup (B \cap A \cap A^f) = (A \cap B^b) \cup (B \cap A^b)$.

4) S_f *implies* III_b.

Proof: We have

$$\begin{aligned} M^c \cap N^c \cap (M^c \cap N^c)^f &= M^c \cap N^c \cap (M^f \cup N^f), (I_f), (II_f), \text{ or} \\ M \cup N \cup (M^c \cap N^c)^{fc} &= M \cup N \cup (M^{fc} \cap N^{fc}). \end{aligned}$$

Intersecting both sides with $M^c \cap N^c$, we obtain:

$$\begin{aligned} M^c \cap N^c \cap (M^c \cap N^c)^{fc} &= M^c \cap N^c \cap (M^{fc} \cap N^{fc}), \text{ or} \\ M \cup N \cup (M \cup N)^f &= M \cup N \cup (M^f \cup N^f). \end{aligned}$$

We deduce easily that $M \subset N$ implies $M^f \subset (N \cup N^f)$.

Now set $M = A^c \cap A^f$ and $N = A^f$. This results in $(A^c \cap A^f)^f \subset (A^f \cup A^{ff}) = A^f$, from which it follows that $(A^c \cap A^f)^f \cap A^{fc} = \varnothing$.

We obtain, taking into account the last relation, $A^{cbcb} = (A^c \cap A^{cf})^c \cap (A^c \cap A^{cf})^{cf} = (A \cup A^{fc}) \cap (A^c \cap A^f)^f = \big(A \cap (A^c \cap A^f)^f\big) \cup \big(A^{fc} \cap (A^c \cap A^f)^f\big) = A \cap (A^c \cap A^f)^f \subset A$. QED

5) S_b *implies* I_r.

Proof: We have $A \cup (A^c \cap B^{cb}) = A \cup (A \cap B^{cb}) \cup (A^c \cap B^{cb}) = A \cup B^{cb}$ and similarly $B \cup (B^c \cap A^{cb}) = B \cup A^{cb}$.

Based on the last two formulas we obtain $(A \cup B)^r = A \cup B \cup (A \cup B)^{cb} = A \cup B \cup (A^c \cap B^c)^b = A \cup B \cup (A^c \cap B^{cb}) \cup (B^c \cap A^{cb}) = A \cup A^{cb} \cup B \cup B^{cb} = A^r \cup B^r$. QED

6) S_b *implies* IV_r.

Proof: Based on (I_b) and (III_b) we obtain $A^{rr} = A \cup A^{cb} \cup (A \cup A^{cb})^{cb} = A \cup A^{cb} \cup (A^c \cap A^{cbc})^b = A \cup A^{cb} \cup (A^c \cap A^{cbcb}) \cup (A^{cb} \cap A^{cbc}) = A \cup A^{cb} = A^r$. QED

§ 2. Basic Properties of the Sets A^e, A^i, A^f, and A^b

Note. We omit formulas that are obtainable from the theorems below via simple applications of the algebra of logic. For example if $A \subset B$, then $A \cup B = B$ and $A \cap B = A$, so it follows from Theorem 1_e that $B^e \subset A^e$. It follows in the same way from Theorem 1_i that $A \subset B$ implies $A^i \subset B^i$.

1. We will prove each of the following theorems:

1_e: $(A \cup B)^e \subset A^e \cup B^e \subset (A \cap B)^e$
2_e: $A^e \setminus B^e \subset (A \setminus B)^e$
3_e: $A^{ec} \cap B^{ce} \subset (A \cap B)^{ec}$
4_e: $A^{ce} \subset A^{ec}$
5_e: $A^{eeee} = A^{ee}$

1_i: $(A \cap B)^i \subset A^i \cup B^i \subset (A \cup B)^i$
2_i: $A^{ic} \cap B^{ci} \subset (A \cup B)^{ic}$
3_i: $A^{ci} \subset A^{ic}$
4_i: $(A \setminus B)^i \subset A^i \setminus B^i$
5_i: $A^{icicici} = A^{ici}$

1_f: $(A \cap B)^f \subset (A \cup B)^f \cup \big[(A \cup B) \cap (A^f \cup B^f)\big] = A^f \cup B^f$
2_f: $A^{fff} = A^{ff}$

1_b: $A \subset B$ implies $A \cap B^b \subset A^b$
2_b: $A^{cbcb} \subset A^b \subset A \subset A^{cbc} \subset A^{bcbc}$
3_b: $A^{bb} = A^b$
4_b: $(A \cup B)^b \subset A^b \cup B^b$
5_b: $A^b \cap B^b \subset (A \cap B)^b \subset A^b \cup B^b$.

2. Here are proofs of the theorems involving the exterior operation:

1_e: Since $M \subset N$ implies $N^e \subset M^e$, we have $(A \cup B)^e \subset A^e$ and $(A \cup B)^e \subset B^e$, hence $(A \cup B)^e \subset A^e \cup B^e$. Similarly $A^e \subset (A \cap B)^e$ and $B^e \subset (A \cap B)^e$, hence $A^e \cup B^e \subset (A \cap B)^e$.

2_e: We have $A^e \subset (A \cap B^c)^e$ by 1_e and thus, a fortiori, $A^e \cap B^{ec} \subset (A \cap B^c)^e$. QED

3_e: Clearly $A \subset (A \cap B) \cup B^c$, thus $\big[(A \cap B) \cup B^c\big]^e \subset A^e$ and $(A \cap B)^e \cap B^{ce} \subset A^e$ (by I_e). Thus $A^{ec} \subset (A \cap B)^{ec} \cup B^{cec}$. Intersecting both sides with B^{ec} yields $A^{ec} \cap B^{ce} \subset (A \cap B)^{ec} \cap B^{ce} \subset (A \cap B)^{ec}$.

4_e: It follows from Axiom II_e that $C^e = \varnothing$. We thus have $A^e = C \cap A^e = C^{ec} \cap A^{cce} \subset (C \cap A^c)^{ec} = A^{cec}$, from which we get $A^{ce} \subset A^{ec}$.

5_e: Using Axiom II_e we get $A^{ee} \subset A^{ec}$. Hence $A^{ece} \subset A^{eee}$. Thus Axiom IV_e yields $A^{eeee} \subset A^{ecee} = A^{ee}$. On the other hand, Axiom II_e implies $A^{eee} \subset A^{eec}$. Thus by Axiom IV_e we get $A^{ee} = A^{eece} \subset A^{eeee}$. Therefore $A^{eeee} = A^{ee}$.

Remark. Theorem 5_e can be replaced by either of the formulas $A = A^{ee}$ or $A^e = A^{eee}$. The condition $A = A^{ee}$ (or $A = A^{cici}$) defines the regular open sets. The condition $A^e = A^{eee}$ (or $A^{ci} = A^{cicici}$) characterizes sets whose closure is regular closed.

Note. A set A is called (by Kuratowski) *regular open* when it equals the interior of its closure. When it equals the closure of its interior it is called *regular closed*.

3. Here are proofs of the theorems involving the interior operation:

1_i: This theorem is easily deduced from the obvious formulas $A \cap B \subset A$, $A \cap B \subset B$, $A \subset A \cup B$, $B \subset A \cup B$ and the fact that $M \subset N$ implies $M^i \subset N^i$.

2_i: We have the obvious relation $(A \cup B) \cap B^c \subset A$. Thus $\big[(A \cup B) \cap B^c\big]^i \subset A^i$. By Axiom I_i, $(A \cup B)^i \cap B^{ci} \subset A^i$. Hence $A^{ic} \subset (A \cup B)^{ic} \cup B^{cic}$. Intersecting both sides with B^{ci} yields $A^{ic} \cap B^{ci} \subset B^{ci} \cap (A \cup B)^{ic} \subset (A \cup B)^{ic}$.

3_i: By Axiom II_i we have $\varnothing^i = \varnothing$. Using Theorem 2_i it follows that $A^{ci} = \varnothing^{ic} \cap A^{ci} \subset A^{ic}$.

4_i: By Theorem 3_i we have $A^i \cap B^{ci} \subset A^i \cap B^{ic}$. By Axiom I_i we have $A^i \cap B^{ci} = (A \cap B^c)^i$. Therefore $(A \cap B^c)^i \subset A^i \cap B^{ic}$. QED

5_i: By Axiom II_i we have $A^{ici} \subset A^{ic}$. Thus $A^i \subset A^{icic}$. By Axiom IV_i it follows that $A^i \subset A^{icici}$. Hence $A^{icicic} \subset A^{ic}$. Thus $A^{icicici} \subset A^{ici}$. On the other hand by Axiom II_i we have $A^{icici} \subset A^{icic}$. This implies $A^{ici} \subset A^{icicic}$. Thus by Axiom IV_i we get $A^{ici} \subset A^{icicici}$. Conclude $A^{icicici} = A^{ici}$.

4. Here are proofs of the theorems involving the frontier operation:

1_f: By Axioms I_f and II_f we have $A^c \cap B^c \cap (A^c \cap B^c)^f = A^c \cap B^c \cap (A^f \cup B^f)$. From De Morgan's laws and Axiom II_f we get the equation $A \cup B \cup (A \cup B)^{fc} = A \cup B \cup (A^{fc} \cap B^{fc})$. Intersecting both sides with $A^c \cap B^c$ gives us $A^c \cap B^c \cap (A \cup B)^{fc} = A^c \cap B^c \cap A^{fc} \cap B^{fc}$. Hence

$$A \cup B \cup (A \cup B)^f = A \cup B \cup A^f \cup B^f. \qquad (\alpha)$$

Consider the identity

$$(A \cup B)^f = \big\{A \cup B \cup (A \cup B)^f\big\} \cap \big\{(A^c \cap B^c) \cup (A \cup B)^f\big\}.$$

From formula (α) we obtain

$$(A \cup B)^f = \big\{A \cup B \cup A^f \cup B^f\big\} \cap \big\{(A^c \cap B^c) \cup (A \cup B)^f\big\}.$$

Unioning both sides with $(A \cup B) \cap (A^f \cup B^f)$ and noting that $A \cup B \cup (A^c \cap B^c) = C$, we get

$$\begin{aligned}(\beta)\qquad &(A \cup B)^f \cup \big\{(A \cup B) \cap (A^f \cup B^f)\big\} = \\ &\big\{(A^f \cup B^f) \cap \big[A \cup B \cup (A^c \cap B^c) \cup (A \cup B)^f\big]\big\} \cup \big[(A \cup B) \cap (A \cup B)^f\big] = \\ &A^f \cup B^f \cup \big[(A \cup B) \cap (A \cup B)^f\big].\end{aligned}$$

From Axiom I_f it follows that $M \subset N$ implies $M \cap N^f \subset M^f$. Hence $A \cap (A \cup B)^f \subset A^f$ and $B \cap (A \cup B)^f \subset B^f$. It follows from these identities that the right hand side of equation (β) equals $A^f \cup B^f$. We conclude that the second part of Theorem 1_f holds.

An immediate consequence of this identity is the inclusion $(A \cup B)^f \subset A^f \cup B^f$, established by Janiszewski [**1,** 20]. Applying Axiom II_f to this inclusion with A^c and B^c in place of A and B yields the first part of Theorem 1_f.

Remark. The analogous formula $(A \cup B)^{ff} \subset A^{ff} \cup B^{ff}$ does not hold in general. Indeed let $R_1 = \mathbb{Q} \cap (1, 2)$, $R_2 = \mathbb{Q} \cap (2, 3)$, $I = (1, 2) \setminus \mathbb{Q}$, $A = I \cup R_2$, and $B = R_1 \cup R_2$. Letting A^f denote the usual frontier operation on the real line it is easy to see that $(A \cup B)^{ff} = \{1, 2, 3\}$ and $A^{ff} \cup B^{ff} = \{1, 3\}$.

2_f: Applying Axiom I_f with A^{ff} and A^f in place of A and B we get $A^{ff} \cap A^f \cap (A^{ff} \cap A^f)^f = A^{ff} \cap A^f \cap (A^{fff} \cup A^{ff})$. By Axiom IV_f we have $A^{ff} \subset A^f$ and $A^{fff} \subset A^{ff}$. Therefore $A^{fff} = A^{ff}$.

Remark. Each of the relations $A = A^f$ and $A = A^{ff}$ implies that $A = A^f = A^{ff}$. The relation $A = A^f$ (and $A = A^{ff}$) holds if and only if A is a closed *frontier set.*

Note. The set A is closed when $A^f \subset A$. It is a frontier set when $A \subset A^f$.

With regard to the relation $A^f = A^{ff}$ we prove the following theorem:

3_f: The relation $A^f = A^{ff}$ holds if and only if the border of A is nowhere dense.

Note. The set A is *nowhere dense* when the interior of its closure is empty. We can express this condition by the formula $A^{ee} = \varnothing$ or by the equivalent formula $(A \cup A^f)^f = A \cup A^f$. It is easily seen that a closed frontier set is always nowhere dense.

Proof of 3_f:

I. Suppose first $A^{ff} = A^f$. The set A^f is always closed (since $A^{ff} \subset A^f$). But by hypothesis A^f is a frontier set (since $A^f \subset A^{ff}$). Hence it is nowhere dense. So we have $A^{fee} = \varnothing$. But $A^b \subset A^f$, from which $A^{fe} \subset A^{be}$ and $A^{bee} \subset A^{fee} = \varnothing$, so A^b is also nowhere dense.

II. Now assume $A^{bee} = \varnothing$.

α) We have $A^{bi} = (A \cap A^{ic})^i = A^i \cap A^{ici} = \varnothing$.

It is easily seen that the formula $M^i = \varnothing$ is equivalent to the formula $M \subset M^f$. Thus $A^b \subset A^{bf}$. But we have by assumption $(A^b \cup A^{bf})^f = A^b \cup A^{bf}$ (see the note above). Hence $A^{bff} = A^{bf}$. It follows that A^{bf} is nowhere dense.

β) We have $A^{icici} \cap A^{ici} \subset A^{icic} \cap A^{ici} = \varnothing$. From this we get $A^{iff} = A^{ifcic} \cap A^{ific} = A^{ifcic} \cap (A^{icic} \cap A^{ic})^{ic} = A^{ifcic} \cap (A^{icici} \cap A^{ici})^c = A^{ifcic}$. But $M \subset M^{cic}$. Hence $A^{if} \subset A^{iff}$. Therefore $A^{iff} = A^{if}$. The set A^{if} is therefore nowhere dense (for all A).

γ) With the sets A^{bf} and A^{if} both being nowhere dense, so is their union.

Note. In [**2**] Kuratowski proves that the union of two nowhere dense sets is nowhere dense.

We claim that $A^{bf} \cup A^{if} = A^f$. We have $A^{bf} = (A \cap A^f)^f \subset A^f \cup A^{ff} = A^f$ and $A^{if} = (A \cap A^{fc})^f \subset A^f \cup A^{ff} = A^f$. Thus $A^{bf} \cup A^{if} \subset A^f$.

On the other hand $A^f = (A^i \cup A^b)^f \subset A^{if} \cup A^{bf}$. Therefore $A^{bf} \cup A^{if} = A^f$, as claimed. It follows that A^f is nowhere dense. Hence $(A^f \cup A^{ff})^f = A^f \cup A^{ff}$. Therefore $A^{ff} = A^f$. QED

5. Here are proofs of the theorems involving the border operation:

1_b: When $A \subset B$, we obtain from Axiom I_b that $A^b = (A \cap B^b) \cup (B \cap A^b)$. Therefore $A \cap B^b \subset A^b$.

2_b:
α) The relation $A^b \subset A$ results from Axiom I_b when we set $B = A$.
β) Hence $A^{cb} \subset A^c$. Thus $A \subset A^{cbc}$.
γ) The inclusion $A \subset A^{cbc}$ implies by Theorem 1_b that $A \cap A^{cbcb} \subset A^b$. Hence by Axiom III_b we get $A^{cbcb} \subset A^b$.
δ) We have $A^{cbcb} \subset A^b$. Substituting A^c for A yields $A^{bcb} \subset A^{cb}$. Therefore $A^{cbc} \subset A^{bcbc}$.
This completes the proof of Theorem 2_b.

3_b: We have $A^b \subset A$. Hence by Theorem 1_b, we get $A^b \subset A^{bb}$. But Theorem 2_b implies $A^{bb} \subset A^b$. Therefore $A^{bb} = A^b$.

4_b: This theorem follows from Theorem 1_b and the obvious formulas $A \subset A \cup B$ and $B \subset A \cup B$.

5_b: This theorem follows from Theorem 1_b, Axiom I_b, and the formulas $A \cap B \subset A$ and $A \cap B \subset B$.

§ 3. Sets Derived from A^c and One of the Operations A^e, A^i, A^f, A^b

1. In this section we study the functions obtainable from any set A by combining the operation A^c and one of the operations A^e, A^i, A^f, and A^b. We calculate in particular the number of distinct functions that can arise and present all inclusions that hold amongst them in general.

2. As for the operations A^e and A^i, it can be shown using our axioms that there exist a maximum of 14 distinct sets obtainable from an arbitrary set A by combining the operations A^c and A^e (respectively A^i). The table below shows all inclusions that hold in general amongst the 14 possible sets (the table for A^i is similar since $A^e = A^{ci}$):

$$\begin{array}{ccccccccc} & & A^{ceee} & \subset & A^{ee} & \subset & A^{eeec} & & \\ A^{ce} & \subset & A^{ceee} & \subset & A^{ceec} & \subset & A^{eeec} & \subset & A^{ec} \\ & & A^{ce} & \subset & A & \subset & A^{ec} & & \\ & & A^{e} & \subset & A^{c} & \subset & A^{cec} & & \\ A^{e} & \subset & A^{eee} & \subset & A^{eec} & \subset & A^{ceeec} & \subset & A^{cec} \\ & & A^{eee} & \subset & A^{cee} & \subset & A^{ceeec} & & \end{array} \qquad (T)$$

Since $A^r = A^{ec}$ and $A^r = A^{cic}$, the above inclusions (and similar ones for A^i) follow immediately from inclusions established by Kuratowski **[2]** involving the operation A^r. Hence we omit the proofs.

To cite a simple example showing that all sets in table (T) are distinguishable and that no further inclusions exist in general, let C be the set of real numbers and $A = [0,1) \cup (1,2) \cup \left[\mathbb{Q} \cap [3,4)\right] \cup \{5\}$.

Note. Kuratowski **[2]** used a slightly different argument to obtain this last result.

The above example will be useful later.

3. Given Axiom II_f, Theorem 2_f, and the principle of double negation, it is easily shown that any function obtained from a set A via the operations A^c and A^f must be identical to one of the following six functions: A, A^f, A^{ff}, A^c, A^{fc}, A^{ffc}.

Defining frontier in the usual way on the real line, the example above distinguishes all six sets. It also shows that the inclusions $A^{ff} \subset A^f$ and $A^{fc} \subset A^{ffc}$ (which hold by Axiom IV_f and the principle of contraposition) are the only inclusions that hold in general among the six sets.

4. The set A^{cbcb} is the (first) residue of A (in the sense of Hausdorff). We know that there exist sets whose family of residues is transfinite. Hence it is possible to generate an infinite number of distinct set functions by combining the operations A^c and A^b.

For $n \geq 1$ and any set A define $\sigma_1(A) = A$ and $\sigma_{n+1}(A) = \sigma_n(A^c)^{bc}$. We shall prove the following theorem:

6_b: For every set A we have the following two doubly infinite sequences of inclusions:

$$\begin{array}{l} \ldots \subset \sigma_4(A)^b \subset \sigma_3(A)^b \subset \sigma_2(A)^b \subset \sigma_1(A)^b \subset \sigma_1(A) \subset \sigma_2(A) \subset \sigma_3(A) \subset \sigma_4(A) \subset \ldots \\ \ldots \subset \sigma_4(A^c)^b \subset \sigma_3(A^c)^b \subset \sigma_2(A^c)^b \subset \sigma_1(A^c)^b \subset \sigma_1(A^c) \subset \sigma_2(A^c) \subset \sigma_3(A^c) \subset \sigma_4(A^c) \subset \ldots \end{array}$$

Note. Here are some terms of the above sequences:

$$\begin{array}{l} \ldots \subset A^{cbcbcbcb} \subset A^{bcbcb} \subset A^{cbcb} \subset A^b \subset A \subset A^{cbc} \subset A^{bcbc} \subset A^{cbcbcbc} \subset \ldots \\ \ldots \subset A^{bcbcbcb} \subset A^{cbcbcb} \subset A^{bcb} \subset A^{cb} \subset A^c \subset A^{bc} \subset A^{cbcbc} \subset A^{bcbcbc} \subset \ldots \end{array}$$

Proof: We show first that $\sigma_n(A) \subset \sigma_{n+1}(A)$ for $n \geq 1$. Proof is by induction on n. The inclusions for $n = 1$ and $n = 2$ have already been shown (Theorem 2_b). We proceed to show that $\sigma_{m-1}(A) \subset \sigma_m(A)$ and $\sigma_m(A) \subset \sigma_{m+1}(A)$ together imply that $\sigma_{m+1}(A) \subset \sigma_{m+2}(A)$.

By Axiom III_b we have $\sigma_{m-1}(A^c)^{bcbcb} \subset \sigma_{m-1}(A^c)^b$. By Theorem 2_b and the induction hypothesis we get

α) $\quad \sigma_{m-1}(A)^b \subset \sigma_{m-1}(A) \subset \sigma_m(A) = \sigma_{m-1}(A^c)^{bc}$.

Since inclusion α) holds for all sets A, it is also true for A^c. Hence $\sigma_{m-1}(A^c)^b \subset \sigma_{m-1}(A)^{bc}$. Thus $\sigma_{m-1}(A^c)^{bcbcb} \subset \sigma_{m-1}(A)^{bc}$. Apply the definition of $\sigma_m(A)$ to get

β) $\quad \sigma_m(A)^{bcb} \subset \sigma_{m-1}(A^{cc})^{bc} = \sigma_m(A^c)$.

By the induction hypothesis we have $\sigma_m(A^c) \subset \sigma_{m+1}(A^c) = \sigma_m(A)^{bc}$. It follows from inclusion β) and Theorem 1_b that $\sigma_m(A)^{bcb} = \sigma_m(A^c) \cap \sigma_m(A)^{bcb} \subset \sigma_m(A^c)^b$. Thus $\sigma_m(A^c)^{bc} \subset \sigma_m(A)^{bcbc}$. But $\sigma_m(A^c)^{bc} = \sigma_{m+1}(A)$ and $\sigma_m(A)^{bcbc} = \sigma_m(A^{cc})^{bcbc} = \sigma_{m+1}(A^c)^{bc} = \sigma_{m+2}(A)$. Conclude $\sigma_{m+1}(A) \subset \sigma_{m+2}(A)$.

The formula $\sigma_n(A) \subset \sigma_{n+1}(A)$ is thus proved. We deduce by contraposition that $\sigma_{n+1}(A)^c \subset \sigma_n(A)^c$. Substituting A^c for A in these last two formulas, we get $\sigma_n(A^c) \subset \sigma_{n+1}(A^c)$ and $\sigma_{n+1}(A^c)^c \subset \sigma_n(A^c)^c$. But $\sigma_n(A)^c = \sigma_{n-1}(A^c)^{bcc} = \sigma_{n-1}(A^c)^b$ and $\sigma_n(A^c)^c = \sigma_{n-1}(A^{cc})^{bcc} = \sigma_{n-1}(A)^b$. All remaining inclusions follow. This completes the proof of Theorem 6_b.

It is easy to see that each set obtained from A via the operations A^c and A^b belongs to one of the two sequences in Theorem 6_b and no further inclusions exist among these sets in general.

§ 4. Applications to Well-Ordered Sets

1. It can be interesting to study the systems of axioms S_r, S_e, S_i, S_f and S_b (and their consequences) when applied to certain well-known spaces. Next we shall examine how they interact with the concept of the residue of a well-ordered set.

Let C be a well-ordered set and A be a subset of C. Define the closure of A as the *residue* of A, namely, the complement of the set of elements that are less than the least element of A. It is easy to verify that the set A^r so defined satisfies the axioms of the system S_r.

2. To help facilitate the discussion of familiar topological concepts within the context of well-ordered sets, we adopt the following definitions:

A set A is an *initial* set when it contains the least element of C. Otherwise, A is *non-initial.*

The set $A^{s_1} = A^{cbcb}$ is called (according to Hausdorff) the *first residue* of A. The set $A^{s_n} = (A^{s_{n-1}})^{s_1}$ is called the nth *residue* of A.

The set $A^{m_1} = A \cap A^{s_1 c}$ is called the *first portion* of A. The nth *portion* of A is given by $A^{m_n} = (A^{s_{n-1}})^{m_1}$.

Note. For example, when $C = \{1,2,3,...\}$ and $A = \{1,2,7,10,11,12\} \cup \{20,21,22,...\}$, we have $A^{m_1} = \{1,2\}$, $A^{m_2} = \{7\}$, $A^{m_3} = \{10,11,12\}$, $A^{m_4} = \{20,21,22,...\}$, $A^{m_n} = \varnothing$ for $n > 4$, $A^{s_1} = \{7,10,11,12\} \cup \{20,21,22,...\}$, $A^{s_2} = \{10,11,12\} \cup \{20,21,22,...\}$, $A^{s_3} = \{20,21,22,...\}$, and $A^{s_n} = \varnothing$ for $n > 3$.

We can also define the notion of portion as follows:

A^{m_1} is the first portion of A when it satisfies the following four properties: 1) $A^{m_1} \subset A$, 2) A^{m_1} contains the least element of A, 3) A^{m_1} is a difference of two residues, 4) A^{m_1} is the largest set satisfying properties 1–3.

The above example helps to illustrate the following relations which hold for any A:

$$A^{s_n} = A \setminus \bigcup_{i=1}^{n} A^{m_i} \quad (n = 1, 2, 3, \ldots) \text{ and}$$

$$A^{m_n} = A^{s_{n-1}} \setminus A^{s_n} \quad (n = 2, 3, 4, \ldots).$$

3. The following propositions are easily verified:

A is *closed* ($A^r = A$) $\Longleftrightarrow$ A equals its residue.
A is *dense* ($A^e = \varnothing$) $\Longleftrightarrow$ A is an initial set.
A is a *frontier set* ($A^i = \varnothing$) $\Longleftrightarrow$ A is a non-initial set.
A is *open* ($A^b = \varnothing$) $\Longleftrightarrow$ A is an initial set with exactly one nonempty portion.
The *interior* of any initial set A is given by $A^i = A^{m_1}$. The interior of any non-initial set A is empty.
The *exterior* of any set A is the set of elements that are less than the least element of A.
The *frontier* of any initial set A is given by $A^f = A^{m_1 c}$. The frontier of any non-initial set A is given by $A^f = A^r$.
The *border* of any initial set A is given by $A^b = A^{s_1}$. The border of any non-initial set A is given by $A^b = A$.

Two nonempty sets A and B can never be separated in C, thus C is connected. It follows immediately that any residue is both closed and connected.

Note. Two sets A and B are said to be *separated* (according to Mazurkiewicz) when $A \cap B^r = B \cap A^r = \varnothing$.

Excluding operations that are constantly equal to $\varnothing$ or C, the closure and complement operations generate six distinct operations on the power set of C: A, A^r, A^{rc}, A^c, A^{cr}, A^{crc}.

Translator's note. The Kuratowski monoid of C contains eight distinct operations: the above six, $A^{rcr} = C$, and $A^{rcrc} = \varnothing$.

References

[1] Janiszewski Z., *Sur les coupures du plan faites par les continus* (*On cutting the plane with continua*), Prace Matem.-Fiz., **26**(1915), 11–63 (in Polish).

[2] Kuratowski K., *Sur l'opération* $\overline{A}$ *de l'analysis situs* (*On the topological closure operation*), Fund. Math., **3**(1922), 182–199 (in French).