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**Some Properties of the Derived Set Operation in Abstract Spaces**

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**Note.** This English translation of M. O. Зарицький's paper, Деякі Властивості Поняття Похідної Множини В Абстрактних Просторах (in Ukrainian), Nauk. Zap. Ser. Fiz.-Mat., **5**(1947) 22–33, was prepared by Mark Bowron in October 2017.

Georg Cantor, the creator of set theory, considered geometric transformations in terms of arbitrary point sets in Euclidean space during the late 1800s. In 1906, Maurice Fréchet observed that point sets satisfy certain basic properties that do not depend on any special properties of the Euclidean continuum.

Fréchet and others have taken properties of set concepts such as boundary, derived set, closure, etc. and deduced further properties of the resulting "abstract" sets, giving rise to theories of general spaces. For example, depending on which topological concepts are accepted as basic and which properties are selected as axioms, various different "topological" spaces arise.

Some mathematicians (Fréchet, Riesz, Sierpiński and others) have taken the concept of derived set as the basis of the theory of point sets. However, the literature does not yet include a systematic and coherent theory of point sets that is based on properties of the derived set. Thus we will construct a complete theory of abstract spaces taking the derived set concept as basic and certain general set properties as independent axioms. This is part of a larger forthcoming work from which some material has already been published.

## I. BASIC CONCEPTS AND NOTATION

**1.** Let $S$ denote the abstract set under consideration and the letters $A$, $B$, $A_1$, $A_2$, etc. subsets of $S$. Let $A^c$ denote the complement of $A$ in $S$; i.e., $A^c = S - A$.

If for every $A \subset S$ there exists a uniquely defined subset $A^d \subset S$, then Alexandrov calls $S$ a *topological space* and $A^d$ the *derived set* of $A$.

We write $A \subset B$ (or $A \to B$) if $A$ is a subset of $B$. If $A$ is not a subset of $B$, we write $A \not\subset B$.

The symbol $\varnothing$ denotes the empty set. Composition of operations shall be denoted by $A^{cd} = (A^c)^d$, $A^{dd} = (A^d)^d$, $A^{dcd} = ((A^d)^c)^d$, etc.

**2.** If $S$ is a topological space, then $A$ will be called:

1) *closed*, if $A^d \subset A$;
2) *dense in itself*, if $A \subset A^d$;
3) *perfect*, if $A^d = A$;
4) *open*, if $A \subset A^{cdc}$;
5) *isolated*, if $A^d \subset A^c$;
6) *everywhere dense*, if $A^d = S$;
7) *nowhere dense*, if $A^{dcd} = S$;
8) *boundary*, if $A \subset A^{cd}$.

**3.** The above definitions imply theorems that hold in every topological space by the laws of Boolean algebra. Here are a few such theorems:

$\boldsymbol{T_1}$**.** The complement of a closed set is an open set. Conversely, the complement of an open set is a closed set.

$\boldsymbol{T_2}$**.** A set is perfect if and only if it is both closed and dense in itself.

$\boldsymbol{T_3}$**.** In no topological space can a nonempty set be both:

*a*) dense in itself and isolated,
*b*) everywhere dense and isolated, or
*c*) boundary and open.

$\boldsymbol{T_4}$. No proper subset of a topological space can be both closed and everywhere dense.

$\boldsymbol{T_5}$. Every everywhere dense set is dense in itself.

$\boldsymbol{T_6}$. Every topological space is a closed set.

$\boldsymbol{T_7}$. A topological space is dense in itself if and only if it is everywhere dense.

$\boldsymbol{T_8}$. Every dense in itself topological space is a perfect set.

$\boldsymbol{T_9}$. A topological space is an open set if and only if the empty set is perfect.

$\boldsymbol{T_{10}}$. An everywhere dense topological space is nowhere dense if and only if the empty set is everywhere dense.

$\boldsymbol{T_{11}}$. A topological space is a boundary set if and only if the empty set is everywhere dense.

$\boldsymbol{T_{12}}$. The empty set is always dense in itself, open, boundary, and isolated.

$\boldsymbol{T_{13}}$. Suppose every set in a topological space is everywhere dense. Then every set is dense in itself, nowhere dense, and boundary. Furthermore, the only set that is closed or perfect is the whole space, and the only set that is open or isolated is the empty set.

$\boldsymbol{T_{14}}$. Suppose every set in a topological space is perfect. Then every set is closed, dense in itself, and open. Furthermore, the only set that is isolated, nowhere dense, or boundary is the empty set, and the only set that is everywhere dense is the whole space.

## II. SYSTEM OF AXIOMS

**1.** To each set $A$ in a topological space there corresponds a derived set $A^d$. This correspondence has been completely arbitrary so far, but now suppose the derived set $A^d$ satisfies the following conditions:

$\boldsymbol{I_d}$ If $A \subset B$, then $A^d \subset B^d$.

$\boldsymbol{II_d}$ $(A \cup B)^d \subset A^d \cup B^d$.

$\boldsymbol{III_d}$ $S^d = S$.

$\boldsymbol{IV_d}$ $A^{dd} \subset A^d$.

$\boldsymbol{V_d}$ $\varnothing^d = \varnothing$.

$\boldsymbol{VI_d}$ If $A^d \subset A \subset A^{cdc}$, then $A = \varnothing$ or $A = S$.

Any space in which the derived set satisfies these conditions will be called a $d$-space.

Conditions $I_d$–$VI_d$ will be referred to as the axioms for a $d$-space.

To prove the independence of the axioms, suppose $S = \{a, b, c\}$. Each column in the following table defines $A^d$ so every axiom holds except for the one listed in the column heading:

| | $\boldsymbol{I_d}$ | $\boldsymbol{II_d}$ | $\boldsymbol{III_d}$ | $\boldsymbol{IV_d}$ | $\boldsymbol{V_d}$ | $\boldsymbol{VI_d}$ |
|---|---|---|---|---|---|---|
| $\varnothing^d$ | $\varnothing$ | $\varnothing$ | $\varnothing$ | $\varnothing$ | $S$ | $\varnothing$ |
| $\{a\}^d$ | $S$ | $\{a\}$ | $\{a\}$ | $\{b\}$ | $S$ | $\{a\}$ |
| $\{b\}^d$ | $S$ | $\{b\}$ | $\{a\}$ | $S$ | $S$ | $\{b\}$ |
| $\{c\}^d$ | $S$ | $\{c\}$ | $\{a\}$ | $S$ | $S$ | $\{c\}$ |
| $\{a,b\}^d$ | $\{a,b\}$ | $S$ | $\{a\}$ | $S$ | $S$ | $\{a,b\}$ |
| $\{b,c\}^d$ | $S$ | $S$ | $\{a\}$ | $S$ | $S$ | $\{b,c\}$ |
| $\{a,c\}^d$ | $S$ | $S$ | $\{a\}$ | $S$ | $S$ | $\{a,c\}$ |
| $S^d$ | $S$ | $S$ | $\{a\}$ | $S$ | $S$ | $S$ |

**2.** To prove the consistency of the axioms, again let $S = \{a, b, c\}$. Set $\varnothing^d = \varnothing$ and $A^d = S$ for $A \neq \varnothing$. It is easy to check that Axioms $I_d$–$VI_d$ all hold.

**3.** If $S = \{a, b\}$, then Axioms $III_d$ and $V_d$ together imply Axiom $I_d$: if $\varnothing^d = \varnothing$ and $S^d = S$, then no matter how $\{a\}^d$ and $\{b\}^d$ are defined, $A \subset B$ implies $A^d \subset B^d$.

**4.** Our system of axioms imposes no condition on the elements of the space. For example, the expression $\{a\} \subset A$, which is equivalent to the statement "the point $a$ is an element of the set $A$," does not appear.

In order to simplify the definitions of basic topological concepts and the proofs of theorems that follow, we express set properties solely in terms of the original sets and the symbols of Boolean algebra.

**5.** Our system of axioms does not require the derived set of a singleton to be empty. Hence our $d$-space differs from the so-called "accessible" space of Fréchet.

## III. CONCLUSIONS FROM AXIOMS $I_d$–$IV_d$

**1.** The following set properties follow from Axioms $I_d$–$IV_d$ and the laws of Boolean algebra:

$\mathbf{1_d}$: $(A \cup B)^d = A^d \cup B^d$.
$\mathbf{2_d}$: $(A \cap B)^d \subset A^d \cap B^d$.
$\mathbf{3_d}$: $A^d \setminus B^d \subset (A \setminus B)^d$.
$\mathbf{4_d}$: $A^{cdc} \subset A^d$.
$\mathbf{5_d}$: $A^{dcdcdcd} = A^{dcd}$.
$\mathbf{6_d}$: If $A \subset B$, then $A^{cdc} \subset B^{cdc}$.
$\mathbf{7_d}$: $A^{dcdd} = A^{dcd}$.
$\mathbf{8_d}$: $A^{ddcd} = A^{dcd}$.

**2.** Before discussing these theorems, we define (in terms of $d$) some basic concepts from the theory of point sets:

| name | symbol | definition |
|---|---|---|
| *closure* | $A^r$ | $A \cup A^d$ |
| *interior* | $A^i$ | $A \cap A^{cdc}$ |
| *exterior* | $A^e$ | $A^c \cap A^{dc}$ |
| *boundary* | $A^j$ | $(A \cap A^{cd}) \cup (A^c \cap A^d)$ |
| *border* | $A^b$ | $A \cap A^{cd}$ |

**3.** The well-known results $1_d$–$3_d$ do not require any discussion. Note that Theorem $4_d$ says more than the well-known inclusion $A^i \subset A$ does, as not only is $A \cap A^{cdc}$ a subset of $A^d$, so is the whole set $A^{cdc}$. It is not difficult to find an example of a set $A$ for which $A \cap A^{cdc} \neq A^{cdc}$. With $S = \mathbb{R}$ take $A = \mathbb{R} \setminus \{0\}$ and define the derived set as the set of accumulation points. Then $A \cap A^{cdc} = A$ and $A^{cdc} = S$.

Let $\varrho$ denote an arbitrary nonempty finite sequence of the operators $d$ and $c$ such that $d$ occurs in the odd positions and $c$ occurs in the even positions (for example $dcd$, $dcdcd$, etc.). The following question is then answered by Theorem $5_d$ (where $A^\varrho$ is obtained from $A$ by applying the composition of operations represented by $\varrho$): For which $\varrho_1 \neq \varrho_2$ does $A^{\varrho_1} = A^{\varrho_2}$ for all $A$? The theorem says that when applied to any set $A$, the sevenfold iteration $dcdcdcd$ always produces the same set as the one produced by the triple iteration $dcd$. Hence there are only six distinct sets of type $A^\varrho$: $A^d$, $A^{dc}$, $A^{dcd}$, $A^{dcdc}$, $A^{dcdcd}$, $A^{dcdcdc}$.

Theorem $6_d$ generalizes the monotonicity of the interior operator ($A \subset B$ implies $A^i \subset B^i$).

Theorem $7_d$ says that $A^{dcd}$ is perfect for all $A$. The sets $A^d$, $A^{dd}$, $A^{ddd}$, etc. are in general distinct from each other, but the presence of the operator $c$ between the first two operators $d$ nullifies this behavior, for applying $d$ to $A^{dcd}$ (repeatedly) produces no new sets.

Theorem $8_d$ says that the operator $dcd$ produces the same set regardless of whether it is applied to $A$ or its derived set $A^d$.

**4.** Let $\sigma$ denote an arbitrary nonempty finite sequence of the operators $d$ and $c$. The set $A^\sigma$ will be called a set of type $A^\sigma$. We consider the following two problems:

**a)** Identify all possible sets of type $A^\sigma$.

**b)** Identify all possible inclusions $A^{\sigma_1} \subset A^{\sigma_2}$ among sets of type $A^\sigma$.

These problems lead us to the following two tables (where $n$ is an arbitrary integer, $A \to B$ means $A \subset B$, and we write $A^{d^2}$ for $A^{dd}$, $A^{d^3}$ for $A^{ddd}$, etc.):

$\boldsymbol{T_{d,1}}$

$$\begin{tikzcd}[column sep=small, row sep=tiny]
 & & & & A^{cdcd} \arrow[dr] & & & & & \\
A^{cdc} \arrow[r] & A^{cddc} \arrow[r] & \cdots \arrow[r] & A^{cd^n c} \arrow[r] & A^{cdcdcdc} \arrow[ur] \arrow[dr] & A^{dcdcd} \arrow[r] & A^{d^n} \arrow[r] & \cdots \arrow[r] & A^{dd} \arrow[r] & A^d \\
 & & & & A^{dcdc} \arrow[ur] & & & & &
\end{tikzcd}$$

$\boldsymbol{T_{d,2}}$

$$\begin{tikzcd}[column sep=small, row sep=tiny]
 & & & & A^{dcd} \arrow[dr] & & & & & \\
A^{dc} \arrow[r] & A^{ddc} \arrow[r] & \cdots \arrow[r] & A^{d^n c} \arrow[r] & A^{dcdcdc} \arrow[ur] \arrow[dr] & A^{cdcdcd} \arrow[r] & A^{cd^n} \arrow[r] & \cdots \arrow[r] & A^{cdd} \arrow[r] & A^{cd} \\
 & & & & A^{cdcdc} \arrow[ur] & & & & &
\end{tikzcd}$$

Based on Axioms $I_d$–$IV_d$ (and Theorems $1_d$–$8_d$, which follow from these axioms), we can prove that:

$\alpha$) All sets of type $A^\sigma$ in both tables are in general distinct from each other.

$\beta$) All inclusions in both tables always hold.

$\gamma$) No further inclusions hold in general among these sets.

$\delta$) The tables include all possible sets of type $A^\sigma$; i.e., any set of type $A^\sigma$ is identical (in general) to one of the sets appearing in tables $T_{d,1}$ and $T_{d,2}$.

## IV. RELATIONSHIP BETWEEN DERIVED SET AND CLOSURE

Kuratowski **[1]** defined topological axioms based on the following properties of the closure operator (similar properties of other topological concepts can be found in my papers **[2]**, **[3]**, **[4]**):

$$\begin{array}{rl}
\boldsymbol{I_r} & (A \cup B)^r = A^r \cup B^r. \\
\boldsymbol{II_r} & A \subset A^r. \\
\boldsymbol{III_r} & \varnothing^r = \varnothing. \\
\boldsymbol{IV_r} & A^{rr} = A^r.
\end{array}$$

It can be shown that Axioms $I_r$–$IV_r$ follow from Axioms $I_d$, $II_d$, $IV_d$, and $V_d$.

Based on Axioms $I_r$–$IV_r$, Kuratowski constructed the following two tables of inclusions among sets obtained by iteration of the operations $r$ and $c$ (applied to an arbitrary set $A$):

$\boldsymbol{T_{r,1}}$

$$\begin{tikzcd}[column sep=small, row sep=small]
 & & \boldsymbol{A^{crcr}} \arrow[dr] & & \\
\boldsymbol{A^{crc}} \arrow[r] \arrow[ddrr] & \boldsymbol{A^{crcrcrc}} \arrow[ur] \arrow[dr] & & \boldsymbol{A^{rcrcr}} \arrow[r] & \boldsymbol{A^r} \\
 & & \boldsymbol{A^{rcrc}} \arrow[ur] & & \\
 & & \boldsymbol{A} \arrow[uurr] & &
\end{tikzcd}$$

$\boldsymbol{T_{r,2}}$

$$\begin{tikzcd}[column sep=small, row sep=small]
 & & \boldsymbol{A^{rcr}} \arrow[dr] & & \\
\boldsymbol{A^{rc}} \arrow[r] \arrow[ddrr] & \boldsymbol{A^{rcrcrc}} \arrow[ur] \arrow[dr] & & \boldsymbol{A^{crcrcr}} \arrow[r] & \boldsymbol{A^{cr}} \\
 & & \boldsymbol{A^{crcrc}} \arrow[ur] & & \\
 & & \boldsymbol{A^c} \arrow[uurr] & &
\end{tikzcd}$$

Here are some relationships between sets in tables $T_{d,1}$ and $T_{d,2}$ and the corresponding sets in tables $T_{r,1}$ and $T_{r,2}$:

$$
\begin{array}{rlll}
\mathbf{9}_{\boldsymbol{d}}: A^{crc} & \subset A^{cdc}. & \mathbf{15}_{\boldsymbol{d}}: A^{crcrc} & = A^{cdcdc}. \\
\mathbf{10}_{\boldsymbol{d}}: A^{d} & \subset A^{r}. & \mathbf{16}_{\boldsymbol{d}}: A^{rcrc} & = A^{dcdc}. \\
\mathbf{11}_{\boldsymbol{d}}: A^{rc} & \subset A^{dc}. & \mathbf{17}_{\boldsymbol{d}}: A^{rcrcrc} & = A^{dcdcdc}. \\
\mathbf{12}_{\boldsymbol{d}}: A^{cd} & \subset A^{cr}. & \mathbf{18}_{\boldsymbol{d}}: A^{crcrcrc} & = A^{cdcdcdc}. \\
\mathbf{13}_{\boldsymbol{d}}: A^{rcr} & = A^{dcd}. & \mathbf{19}_{\boldsymbol{d}}: A^{crcrcr} & = A^{cdcdcd}. \\
\mathbf{14}_{\boldsymbol{d}}: A^{crcr} & = A^{cdcd}. & \mathbf{20}_{\boldsymbol{d}}: A^{rcrcr} & = A^{dcdcd}.
\end{array}
$$

We see that some sets in tables $T_{d,1}$ and $T_{d,2}$ are identical with the corresponding sets in Kuratowski's tables. Note, Kuratowski's tables are finite since $A^{rr} = A^r$; tables $T_{d,1}$ and $T_{d,2}$ are infinite because $A^{dd}$ is in general different from $A^d$.

## V. CONCLUSIONS FROM AXIOM $VI_d$

It can be shown that Axiom $VI_d$ implies the space $S$ is coherent (a space is *coherent* if it cannot be decomposed into the union of two nonempty sets $M$ and $N$ such that $(M \cap N) \cup (M \cap N^d) \cup (N \cap M^d) = \varnothing$).

Axioms $III_d$ and $V_d$ imply that the empty set and whole space are both open and closed. Axiom $VI_d$ says that no other sets in a $d$-space can satisfy this property.

## VI. PROPERTIES OF THE SET $A^{dcd}$

**1.** The set $A^{dcd}$ plays an important role in some proofs. Let us denote it by $A^m$. One can prove the following properties of this set:

$$
\begin{array}{rl}
\mathbf{1}_{\boldsymbol{m}}: & \text{If } A \subset B, \text{ then } B^m \subset A^m. \\
\mathbf{2}_{\boldsymbol{m}}: & A^{mmm} = A^{mcm} = A^m. \\
\mathbf{3}_{\boldsymbol{m}}: & \varnothing^m = S. \\
\mathbf{4}_{\boldsymbol{m}}: & S^m = \varnothing. \\
\mathbf{5}_{\boldsymbol{m}}: & (A \cap B)^{mm} \subset A^{mm} \cap B^{mm}. \\
\mathbf{6}_{\boldsymbol{m}}: & A^{mm} \cup B^{mm} \subset (A \cup B)^{mm}. \\
\mathbf{7}_{\boldsymbol{m}}: & A^{mc} \subset A^{mm}. \\
\mathbf{8}_{\boldsymbol{m}}: & (A \cup B)^m \subset A^m \cap B^m. \\
\mathbf{9}_{\boldsymbol{m}}: & A^m \cup B^m \subset (A \cap B)^m. \\
\mathbf{10}_{\boldsymbol{m}}: & A^{cm} \subset A^{mm}. \\
\mathbf{11}_{\boldsymbol{m}}: & \text{if } A \subset B, \text{ then } A^{mm} \subset B^{mm}. \\
\mathbf{12}_{\boldsymbol{m}}: & A^m \subset A^{cmm}. \\
\mathbf{13}_{\boldsymbol{m}}: & A^m \cup A^{mm} = S.
\end{array}
$$

**2.** It can be shown that no set $A$ satisfies $A^m = A$.

We have that $A^m$ is the derived set of the set of external points of $A$. Each of the following conditions is sufficient for $A$ to be everywhere dense: 1) $A^m = \varnothing$, 2) $A^m \subset A$, 3) $A^{mm} = S$.

The sets $A \cap A^{mm}$ and $A \setminus A^{mm}$ are always nowhere dense.

Each of the following conditions is necessary and sufficient for $A$ to be nowhere dense: 1) $A^m = S$, 2) $A^{mm} = \varnothing$, 3) $A \subset A^m$, 4) $A \subset A^{mmc}$, 5) $A^{mm} \subset A^m$, 6) $A^{d^n} \subset A^m$ for some $n \geq 1$.

**3.** We have the following tables of inclusions:

$\boldsymbol{T_{m,1}}$

$$\begin{tikzcd}[row sep=small] & A^{m} \arrow[dr] & \\ A^{mmc} \arrow[ur] \arrow[dr] & & A^{cmm} \\ & A^{cmc} \arrow[ur] & \end{tikzcd}$$

$\boldsymbol{T_{m,2}}$

$$\begin{tikzcd}[row sep=small] & A^{cm} \arrow[dr] & \\ A^{cmmc} \arrow[ur] \arrow[dr] & & A^{mm} \\ & A^{mc} \arrow[ur] & \end{tikzcd}$$

## VII. PROPERTIES OF CONTINUOUS FUNCTIONS AND HOMEOMORPHISMS

**1.** Suppose that to each $a \in S$ a point $a^{\varphi} \in S$ is assigned. The inverse operation is denoted by $a^{\varphi^{-1}}$. The image of $A$ under $\varphi$ is denoted by $A^{\varphi}$. If $A^{\varphi\varphi^{-1}} = A$ for all $A$, then $\varphi$ is called an *injection*.

The function $\varphi$ is called *continuous* if for each $A \subset S$ we have

$$A^{d\varphi} \subset A^{\varphi} \cup A^{\varphi d}. \tag{1}$$

**2.** It can be shown that in Euclidean space, the function $\varphi$ is continuous if and only if

$$A^{d\varphi} \cup A^{d\varphi d} \subset A^{\varphi} \cup A^{\varphi d} \tag{2}$$

for each $A \subset S$.

We will refer to inclusion (2) as the second definition of continuity.

Inclusions (1) and (2) are equivalent definitions of continuity in Euclidean space, but they are not equivalent in a general topological space.

It can be shown that their equivalence is a necessary and sufficient condition for Axioms $I_d$, $II_d$, and $IV_d$ to be satisfied.

**3.** An injection $\varphi$ is called *mutually continuous* if

$$A^{d\varphi} \subset A^{\varphi} \cup A^{\varphi d} \text{ and } A^{d\varphi^{-1}} \subset A^{\varphi^{-1}} \cup A^{\varphi^{-1}d}$$

(the first definition of continuity), or if

$$A^{d\varphi} \cup A^{d\varphi d} \subset A^{\varphi} \cup A^{\varphi d} \text{ and } A^{d\varphi^{-1}} \cup A^{d\varphi^{-1}d} \subset A^{\varphi^{-1}} \cup A^{\varphi^{-1}d}$$

(the second definition of continuity), for each $A \subset S$.

It turns out that each definition of mutual continuity is equivalent to the following two conditions:

$$A^{d\varphi} \subset A^{\varphi d} \text{ and } A^{d\varphi^{-1}} \subset A^{\varphi^{-1}d}.$$

**Translator's note.** If anyone can verify the above statement please email me at mathematrucker@gmail.com.

The function $\varphi$ is called a *homeomorphism* if for each $A \subset S$,

$$A^{d\varphi} = A^{\varphi d}.$$

We give examples showing the roles played by Axioms $I_d$–$VI_d$ in the theory of continuous functions. Recall the following classical theorems:

$\boldsymbol{C_1}$**.** A continuous function of a continuous function is always continuous.

$\boldsymbol{C_2}$**.** The identity transformation is continuous.

$\boldsymbol{C_3}$**.** The set of mutually continuous bijections forms a group under composition.

$\boldsymbol{C_4}$**.** The set of homeomorphisms forms a group under composition.

$\boldsymbol{C_5}$**.** A function is a homeomorphism if and only if it is a mutually continuous bijection.

$\boldsymbol{C_6}$**.** Constant functions are continuous.

These well-known theorems of classical analysis need not hold in more general spaces. In the following paragraphs we summarize which axioms are needed for each to hold in a general topological space.

To obtain $C_1$ axioms $I_d$ and $II_d$ are needed. To obtain $C_2$ under the second definition of continuity, axiom $IV_d$ is needed. Under the first definition, $C_2$ holds with no additional axioms.

To obtain $C_3$, axioms $I_d$, $II_d$ and $IV_d$ are needed.

Theorem $C_4$ holds with no additional axioms.

Under the first definition of continuity, $C_5$ holds with no additional axioms. Under the second definition, axioms $I_d$, $II_d$ and $IV_d$ are needed.

Theorem $C_6$ holds with no additional axioms under both definitions of continuity.

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**Note.** The following summary appeared in Russian in the original.

M. Zarycki

**On Some Properties of the Derived Set Operation in Abstract Spaces**


Summary:

Let $S$ be a topological space in which the derived set $A^d$ of the set $A \subset S$ satisfies the following axioms:

$$
\begin{aligned}
&\textbf{I}.\ A \subset B \text{ implies } A^d \subset B^d,\\
&\textbf{II}.\ (A \cup B)^d \subset A^d \cup B^d,\\
&\textbf{III}.\ S^d = S,\\
&\textbf{IV}.\ A^{dd} \subset A^d,\\
&\textbf{V}.\ \varnothing^d = \varnothing,\\
&\textbf{VI}.\ A^d \subset A \subset A^{cdc} \text{ implies } A = \varnothing \text{ or } A = S,
\end{aligned}
$$

where $A^c$ denotes the complement of $A$ in $S$, namely $S \setminus A$.

It can be proved that Axioms I–VI are mutually independent.

From the above axioms we can deduce the following formulas:

$$
\begin{aligned}
&1)\ (A \cup B)^d = A^d \cup B^d,\\
&2)\ (A \cap B)^d \subset A^d \cap B^d,\\
&3)\ (A \setminus B)^d \subset A^d \setminus B^d,\\
&4)\ A^{cdc} \subset A^d,\\
&5)\ A^{dcdcdcd} = A^{dcd},\\
&6)\ A \subset B \text{ implies } A^{cdc} \subset B^{cdc},\\
&7)\ A^{dcdd} = A^{dcd} = A^{ddcd}.
\end{aligned}
$$

Many other properties of the derived set operation can also be deduced.

If we denote by $A^r$ the closure of $A$, $A^r = A \cup A^d$, then from axioms I–V can be derived the identity $A^{rcr} = A^{dcd}$.

The following theorem also holds. Each of the relations

$$
\begin{aligned}
&1)\ A^{dcd} = S,\\
&2)\ A^{dcdcd} = \varnothing,\\
&3)\ A \subset A^{dcd},\\
&4)\ A \subset A^{dcdcdc},\\
&5)\ A^{dcdcd} \subset A^{dcd},\\
&6)\ A^d \subset A^{dcd},\\
&7)\ A^{d^n} \subset A^{dcd}
\end{aligned}
$$

(where $A^{d^n}$ is the $n$th derived set of $A$) is a necessary and sufficient condition for the set $A$ to be nowhere dense.

Let $\varphi$ be a single-valued function transforming each point of $S$ to some point of $S$ and let $A^\varphi$ denote the set of images of points in $A$ under $\varphi$. In Euclidean space each of the formulas

$$1)\ A^{d\varphi} \subset A^\varphi \cup A^{\varphi d} \text{ and}$$
$$2)\ A^{d\varphi} \cup A^{d\varphi d} \subset A^\varphi \cup A^{\varphi d}$$

is a necessary and sufficient condition for the function $\varphi$ to be continuous. These two definitions are equivalent in spaces where the derived set satisfies axioms I, II and IV. There are also three similar conditions so that an injection $\varphi$ $(A^{\varphi\varphi^{-1}} = A)$ is mutually continuous:

$$1)\ A^{d\varphi} \subset A^\varphi \cup A^{\varphi d} \text{ and } A^{d\varphi^{-1}} \subset A^{\varphi^{-1}} \cup A^{\varphi^{-1}d},$$
$$2)\ A^{d\varphi} \cup A^{d\varphi d} \subset A^\varphi \cup A^{\varphi d} \text{ and } A^{d\varphi^{-1}} \cup A^{d\varphi^{-1}d} \subset A^{\varphi^{-1}} \cup A^{\varphi^{-1}d},$$
$$3)\ A^{d\varphi} \subset A^{\varphi d} \text{ and } A^{d\varphi^{-1}} \subset A^{\varphi^{-1}d}.$$

If $A^{d\varphi} = A^{\varphi d}$, then we have a homeomorphism.

It can be proved that a continuous function of a continuous function is a continuous function in spaces that satisfy Axioms I and II.

The identity transformation is continuous under the first definition in general topological spaces, and second-definition continuous in spaces that satisfy Axioms I, II and IV. Homeomorphisms form a group; constant functions are continuous in general topological spaces. According to the first definition of continuity, homeomorphism is equivalent to mutual continuity and bijectivity in any topological space, but according to the second definition of continuity they are equivalent only in spaces where the derived set satisfies Axioms I, II and IV.

## References


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## Appendix

We verify some results that were stated without proof.

**Section I.3**

**$T_1$.** Suppose $A$ is closed. Then $A^d \subset A$. Taking complements, we get $(A^c)^{cdc} = A^{dc} \supset A^c$. Conclude $A^c$ is open. Now suppose $A$ is open. Then $A \subset A^{cdc}$. Taking complements, we get $A^c \supset A^{cd}$. Conclude $A^c$ is closed.

**$T_2$–$T_8$** follow immediately from the definitions.

**$T_9$.** Suppose $S$ is open. Then $S \subset S^{cdc} = \varnothing^{dc}$. Hence $\varnothing^{dc} = S$. Thus $\varnothing^d = \varnothing$. Conclude $\varnothing$ is perfect. Conversely suppose $\varnothing$ is perfect. Then $S^{cdc} = \varnothing^{dc} = \varnothing^c = S$. Hence $S \subset S^{cdc}$. Conclude $S$ is open.

**$T_{10}$.** Let $S$ be an everywhere dense topological space. Suppose $S$ is nowhere dense. Then $S^{dcd} = S$. Since $S$ is everywhere dense, we have $S^d = S$. Thus $S^{cd} = S$. Hence $\varnothing^d = S$. Conclude $\varnothing$ is everywhere dense. Conversely suppose $\varnothing$ is everywhere dense. Then $S^{cd} = \varnothing^d = S$. Since $S$ is everywhere dense, this implies $S^{dcd} = S$. Conclude $S$ is nowhere dense.

**$T_{11}$.** Suppose $S$ is a boundary set. Then $S \subset S^{cd}$. Hence $S^{cd} = S$. Thus $\varnothing^d = S$. Conclude $\varnothing$ is everywhere dense. Reversing the steps suffices to prove the converse.

**$T_{12}$–$T_{14}$** follow immediately from the definitions.

**Section III.1**

**$1_d$.** By Axiom $II_d$ we need to prove that $A^d \cup B^d \subset (A \cup B)^d$, but this holds by Axiom $I_d$ since $A \subset (A \cup B)$ and $B \subset (A \cup B)$.

**$2_d$.** This follows from Axiom $I_d$ since $(A \cap B) \subset A$ and $(A \cap B) \subset B$.

**$3_d$.** By result $1_d$ we have $A^d = (A \setminus B)^d \cup (A \cap B)^d$. By Axiom $I_d$ we have $(A \cap B)^d \subset B^d$, hence $B^{dc} \subset (A \cap B)^{dc}$. Taken together these imply $A^d \setminus B^d = A^d \cap B^{dc} \subset (A \setminus B)^d$.

**$4_d$.** By result $1_d$ and Axiom $III_d$ we have $S = S^d = (A^c \cup A)^d = A^{cd} \cup A^d$. Hence $A^{dc} \subset A^{cd}$. Therefore $A^{cdc} \subset A^d$.

**$5_d$.** By result $4_d$ and Axiom $IV_d$ we have $A^{dcdc} \subset A^{dd} \subset A^d$. By Axioms $I_d$ and $IV_d$ this implies $A^{dcdcd} \subset A^{dd} \subset A^d$. Hence $A^{dc} \subset A^{dcdcdc}$, which by Axiom $I_d$ implies $A^{dcd} \subset A^{dcdcdcd}$. Similarly, by result $4_d$ and Axiom $IV_d$, we have $A^{dcdcdc} \subset A^{dcdd} \subset A^{dcd}$. By Axioms $I_d$ and $IV_d$ we get $A^{dcdcdcd} \subset A^{dcdd} \subset A^{dcd}$. Therefore $A^{dcdcdcd} = A^{dcd}$.

**$6_d$.** Suppose $A \subset B$. Then $B^c \subset A^c$. Hence by Axiom $I_d$ we get $B^{cd} \subset A^{cd}$. Thus $A^{cdc} \subset B^{cdc}$.

**$7_d$.** By Axiom $IV_d$ we have $A^{dcdd} \subset A^{dcd}$. By result $4_d$ and Axiom $IV_d$ we have $A^{dcdc} = (A^d)^{cdc} \subset (A^d)^d \subset A^d$. Thus $A^{dc} \subset A^{dcd}$. By Axiom $I_d$ this implies $A^{dcd} \subset A^{dcdd}$. Therefore $A^{dcdd} = A^{dcd}$.

**$8_d$.** By Axiom $IV_d$ we have $A^{dd} \subset A^d$. Thus $A^{dc} \subset A^{ddc}$. Hence by Axiom $I_d$ we get $A^{dcd} \subset A^{ddcd}$. On the other hand by $4_d$ we have $A^{dc} \subset A^{cd}$. Since this holds for all $A$, we can substitute $A^d$ for $A$ to get $A^{ddc} \subset A^{dcd}$. By Axiom $I_d$ and result $7_d$, this implies $A^{ddcd} \subset A^{dcdd} = A^{dcd}$. Therefore $A^{ddcd} = A^{dcd}$.

**Section IV**

**$9_d$–$12_d$.** Inclusion $10_d$ follows immediately from the definition of $A^r$ $(A \cup A^d)$; the other three inclusions follow from $10_d$ by applying $c$ to both sides and/or substituting $A^c$ for $A$.

**$13_d$.** Applying $d$ to both sides of $11_d$ gives us $A^{rcd} \subset A^{dcd}$. By $T_{d,1}$ we get

$$A^{rc} = (A \cup A^d)^c = A^c \cap A^{dc} \subset A^{dc} \subset A^{dcd}.$$

Hence $A^{rcr} = A^{rc} \cup A^{rcd} \subset A^{dcd}$.
By $T_{d,2}$ and $12_d$ we have $A^{dcd} \subset A^{cd} \subset A^{cr}$. It follows by $T_{d,2}$ and $1_d$ that

$$A^{dcd} = (A^{dc})^d = (A^{dc} \cap A^{ddc})^d = (A^d \cup A^{dd})^{cd} = (A \cup A^d)^{dcd} = (A^r)^{dcd} \subset (A^r)^{cr} = A^{rcr}.$$

Conclude $A^{rcr} = A^{dcd}$.

**$14_d$–$16_d$** follow from $13_d$ by applying $c$ to both sides and/or substituting $A^c$ for $A$.

**$17_d$–$19_d$** follow from $20_d$ (below) by applying $c$ to both sides and/or substituting $A^c$ for $A$.

**$20_d$.** By $T_{d,1}$ and $16_d$ we have

$$A^{rcrcr} = (A^{rcrc})^r = (A^{dcdc})^r = (A^{dcdc} \cup A^{dcdcd}) = A^{dcdcd}.$$

**Section VI.1**

**$1_m$.** This holds since $d$ is increasing.

**$2_m$.** This follows from $5_d$, $7_d$ and $8_d$.

**$3_m$–$4_m$** are immediate.

**$5_m$.** By $1_d$ and $2_d$ it follows that

$$\begin{array}{lcl} (A \cap B)^d & \subset & A^d \cap B^d \\ (A \cap B)^{dc} & \supset & A^{dc} \cup B^{dc} \\ (A \cap B)^{dcd} & \supset & A^{dcd} \cup B^{dcd} \\ (A \cap B)^{dcdc} & \subset & A^{dcdc} \cap B^{dcdc} \\ (A \cap B)^{dcdcd} & \subset & A^{dcdcd} \cap B^{dcdcd} \end{array}$$

By $7_d$ we conclude $(A \cap B)^{mm} \subset A^{mm} \cap B^{mm}$.

**$6_m$.** By $1_d$ and $2_d$ it follows that

$$\begin{array}{lcl} (A \cup B)^d & = & A^d \cup B^d \\ (A \cup B)^{dc} & = & A^{dc} \cap B^{dc} \\ (A \cup B)^{dcd} & \subset & A^{dcd} \cap B^{dcd} \\ (A \cup B)^{dcdc} & \supset & A^{dcdc} \cup B^{dcdc} \\ (A \cup B)^{dcdcd} & \supset & A^{dcdcd} \cup B^{dcdcd} \end{array}$$

By $7_d$ we conclude $A^{mm} \cup B^{mm} \subset (A \cup B)^{mm}$.

**$7_m$.** This holds by $T_{d,1}$ and $7_d$.

**$8_m$.** See the first inclusion in the proof of $6_m$.

**$9_m$.** See the third inclusion in the proof of $5_m$.

**$10_m$.** This holds by $T_{d,1}$ and $7_d$.

**$11_m$.** This holds since $d$ is increasing.

**$12_m$.** This holds by $T_{d,2}$ and $7_d$.

**$13_m$.** By $T_{d,1}$ and $7_d$ we have $A^{mc} \subset A^{mm}$, hence $A^m \cup A^{mm} = S$.

**Section VI.2**

Note that by $7_m$ we have $(A^m = A) \Rightarrow (A^c \subset A) \Rightarrow (A = S)$. Since $S^m \neq S$, we conclude that no set $A$ satisfies $A^m = A$.

1) Suppose $A^m = \varnothing$. Then $A^{dcdc} = S$. But $A^{dcdc} \subset A^d$; thus $A^d = S$.

2) Suppose $A^m \subset A$. By $7_m$ we get $A^{dcd} = A^{dcdd} \subset A^d$. Hence $A^{dc} \subset A^{dcdc} \subset A^d$. Conclude $A^d = S$.

3) Suppose $A^{mm} = S$. By $7_m$ we get $A^{dcdcd} = S$. By $5_d$ this implies $A^{dcd} = A^{dcdcdcd} = \varnothing$. Conclude by 1) that $A^d = S$.

By $9_m$ and $13_m$ we have $S = (A^m) \cup (A^m)^{mm} \subset (A \cap A^{mm})^m$. Thus $(A \cap A^{mm})^m = S$. Conclude $A \cap A^{mm}$ is nowhere dense.

By $2_m$ and $7_m$ we have $A^{mmc} = (A^m)^{mc} \subset (A^m)^{mm} = A^{mmm} = A^m$. Hence $A \cap A^{mmc} \subset A \cap A^m$. Thus, by $1_m$, $9_m$ and $13_m$, we get $(A \cap A^{mmc})^m \supset (A \cap A^m)^m \supset A^m \cup A^{mm} = S$. Therefore $(A \cap A^{mmc})^m = S$. Conclude $A \setminus A^{mm}$ is nowhere dense.

By $T_{d,1}$ it follows that

$$2)\ A^{mm} = \varnothing \iff 1)\ A^m = S \implies 6)\ A^{d^n} \subset A^m \text{ for some n} \geq 1 \implies 5)\ A^{mm} \subset A^m.$$

But 5) implies 1) by $13_m$. Hence 1), 2), 5) and 6) are equivalent. By $T_{d,2}$ we have $A^{mmc} \subset A^m$. Hence

$$4)\ A \subset A^{mmc} \quad \Longrightarrow \quad 3)\ A \subset A^m.$$

Clearly 2) implies 4). Finally, 3) implies 5) by $1_m$. Conclude all six conditions are equivalent.

**Section VII.2**

Continuity of the second kind obviously implies continuity of the first kind, so we need only prove the converse to get equivalence. To that end, suppose $\varphi$ satisfies

$$A^{d\varphi} \subset A^{\varphi} \cup A^{\varphi d}$$

for each $A \subset S$. Axioms $I_d$, $II_d$ and $IV_d$ imply

$$A^{d\varphi d} \subset (A^{\varphi} \cup A^{\varphi d})^d \subset (A^{\varphi})^d \cup (A^{\varphi d})^d = (A^{\varphi})^d \cup (A^{\varphi})^{dd} = (A^{\varphi})^d \subset A^{\varphi} \cup A^{\varphi d}.$$

Thus $A^{d\varphi} \cup A^{d\varphi d} \subset A^{\varphi} \cup A^{\varphi d}$. Conclude the two definitions of continuity are equivalent.

Conversely, it is clear that if the two definitions are not equivalent, then at least one of the three axioms cannot hold (otherwise we get a contradiction using the same argument as above). Thus we conclude the definitions are equivalent iff Axioms $I_d$, $II_d$ and $IV_d$ hold.

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