A. V. Chagrov
(Kalinin State University)

Kuratowski Numbers

Note. This English translation of A. B. Чагров's paper, Числа Куратовского (in Russian), in *Application of Functional Analysis in Approximation Theory,* Kalinin State Univ., (1982) 186–190, was prepared by Mark Bowron in August 2012.

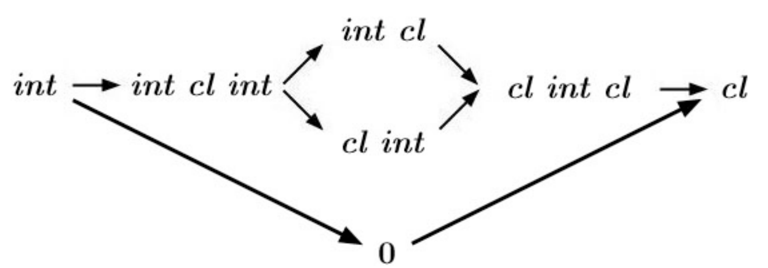
Kuratowski defined a topological space as a set with an operation on its subsets — namely, the closure operation. The closure of A (clA) has the following properties:

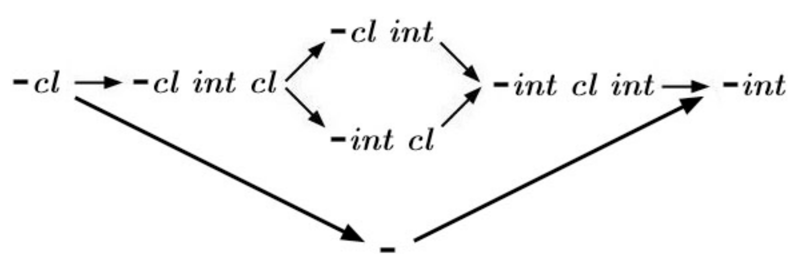
$$cl(A \cup B) = clA \cup clB, \quad A \subseteq clA,$$
$$cl(clA) = clA, \quad cl\varnothing = \varnothing.$$

The interior operation ($intA$) is the dual of the closure operation: $intA = -cl-A$ (where $-A$ denotes the complement of A).

The following "fun and instructive problem" appears in [**1,** 70]: *How many distinct sets can be obtained from a given set by successive application of the closure and complement operations*? As shown by Kuratowski in [**2**], the maximum number of distinct sets obtained in this way is 14.

We shall consider a modification of the above problem. Let T be a topological space. Let the *Kuratowski number* of T (symbolically, $K(T)$) be defined as the number of distinct operations obtained by combining the closure and complement operations in T. It follows from Kuratowski's result that $K(T) \leq 14$. The interrelationships between these operations are displayed in the following diagrams:





Here 0 is the identity operation, i.e., the operation which leaves every set unchanged, and the arrow $f \longrightarrow k$ means that for any set A we have $fA \subseteq kA$. Every operation derived from the operations of closure and complement coincides with at least one of the operations in these two diagrams. No further arrows hold in general. In what follows, operations in the first diagram will be called *positive.* Those in the second will be called *negative.*

Note, the inclusion diagrams for a given topological space can never contain any arrows connecting positive and negative operations. For, the existence of such would imply $intA = \varnothing$ for all $A \subseteq T$. No topological space satisfies this, since $T \neq \varnothing$ and $intT = T$.

Hence, by duality, $K(T)$ is even for every topological space T. A natural question arises: What values can the Kuratowski number take in an arbitrary topological space?

There are spaces for which the Kuratowski number is 14 (see [**1**]). If $K(T) < 14$, then some of the arrows in the inclusion diagrams above must somehow combine to form new arrows. Thus, it suffices to determine which new arrows can actually occur.

It turns out (a simple proof, consisting of an exhaustive treatment of all possibilities, we omit) that any nontrivial combination of arrows must be equivalent to one of the following cases (see reference [**A1**] of the appendix for a full proof):

$$1.\ \mathit{int\ cl} \longrightarrow \mathit{cl\ int}, \quad 2.\ \mathit{cl\ int} \longrightarrow \mathit{int\ cl},$$
$$3.\ \mathit{int\ cl} \longleftrightarrow \mathit{cl\ int}, \quad 4.\ \mathit{cl} \longrightarrow \mathit{int\ cl}, \quad 5.\ \mathit{cl} \longrightarrow 0.$$

These five cases correspond to the following diagrams (by duality we give only the positive operations):

(1)
$$\begin{tikzcd} \mathit{int} \arrow[r] \arrow[drr] & \mathit{int\ cl} \arrow[r] & & \mathit{cl\ int} \arrow[r] & \mathit{cl} \\ & & \mathbf{0} \arrow[urr] & & \end{tikzcd}$$

(2)
$$\begin{tikzcd} \mathit{int} \arrow[r] \arrow[drr] & \mathit{cl\ int} \arrow[r] & & \mathit{int\ cl} \arrow[r] & \mathit{cl} \\ & & \mathbf{0} \arrow[urr] & & \end{tikzcd}$$

(3)
$$\begin{tikzcd} \mathit{int} \arrow[r] \arrow[dr] & \mathit{int\ cl} \arrow[r] & \mathit{cl} \\ & \mathbf{0} \arrow[ur] & \end{tikzcd}$$

(4)
$$\begin{tikzcd} \mathit{int} \arrow[rr] \arrow[dr] & & \mathit{cl} \\ & \mathbf{0} \arrow[ur] & \end{tikzcd}$$

(5)
$$\mathbf{0}$$

Are there spaces in which these diagrams are realized?

EXAMPLE 1. Let $T_1 = \{a, b, c, d\}$ be a topological space with base $\{\varnothing, \{b\}, \{c, d\}, \{a, b, c, d\}\}$. It is easy to verify that $K(T_1) = 14$.

To the author's knowledge, topological spaces corresponding to diagram (1) have not been investigated specifically. In mathematical logic, they are closely connected with calculation based on the law of "weak excluded middle". Hence we shall call such spaces *WEM-spaces*.

EXAMPLE 2. We get a WEM-space $T_2 = \{a, b, c, d, e\}$ by setting the base for the topology equal to $\{\varnothing, \{a\}, \{b\}, \{a, c\}, \{b, d\}, \{a, c, e\}, \{b, d, e\}\}$. Hence $K(T_2) = 10$.

Translator's note. As pointed out in [**A1,** 13], the singleton $\{e\}$ is missing from the base for T_2. After including it, the resulting space satisfies diagram (3). Hence its Kuratowski number is 8. (See the appendix for correct examples.)

Diagram (2) corresponds to extremally disconnected topological spaces. In fact, diagram (2) may be taken as the definition of an extremally disconnected space.

EXAMPLE 3. We get an extremally disconnected space $T_3 = \{a, b, c, d, e\}$ by setting the base for the topology equal to $\{\varnothing, \{a, e\}, \{a, b, e\}, \{a, c, e\}, \{a, b, c, d, e\}\}$. Hence $K(T_3) = 10$.

It is easy to see that diagram (3) is equivalent to diagrams (1) and (2) holding simultaneously. Therefore spaces realizing diagram (3) must be both extremally disconnected and WEM-spaces.

Example 4. We get an extremally disconnected WEM-space $T_4 = \{a, b, c, d\}$ by setting the base for the topology equal to $\{\varnothing, \{a\}, \{a, b\}, \{a, c\}, \{a, c, d\}\}$. Hence $K(T_4) = 8$.

When diagram (4) is realized in a topological space, the system of open (closed) sets forms a Boolean algebra. Examples of this type of space are provided by indiscrete spaces. In view of this, we shall call this type of space *generalized indiscrete*. Note that if a generalized indiscrete space is a WEM-space, then it is trivial, i.e., $cl = 0$. Note also, any generalized indiscrete space is an extremally disconnected space.

Example 5. We get a generalized indiscrete space $T_5 = \{a, b, c\}$ by setting the base for the topology equal to $\{\varnothing, \{a\}, \{b, c\}\}$. Hence $K(T_5) = 6$. Note that T_5 is not indiscrete.

Obviously, diagram (5) corresponds to trivial topological spaces. The following theorems summarize our results.

Theorem 1. The possible Kuratowski numbers are: 14, 10, 8, 6, 2.

Corollary. There are no topological spaces with a Kuratowski number of 12 or 4.

Theorem 2. $K(T) = 2$ if and only if T is a trivial topological space.

Theorem 3. $K(T) = 6$ if and only if T is a nontrivial generalized indiscrete topological space.

Theorem 4. $K(T) = 8$ if and only if T is a nontrivial extremally disconnected WEM-space.

Theorem 5. $K(T) = 10$ if and only if T satisfies precisely one of the following conditions:

a) T is an extremally disconnected space and not generalized indiscrete;
b) T is a WEM-space and not generalized indiscrete.

Theorem 6. $K(T) = 14$ if and only if T is neither extremally disconnected nor a WEM-space.

References

[**1**] Kelley J. L., *General Topology* (Russian translation), Nauka, Moscow, 1968.

[**2**] Kuratowski K., *Sur l'opération $\overline{A}$ de l'analysis situs* (*On the topological closure operation*), Fund. Math., **3**(1922), 182–199 (in French).

Translator's notes. In the original, the names for spaces satisfying diagram 1 (WEM-spaces) and spaces satisfying diagram 2 (extremally disconnected spaces) were mismatched (this observation further explains the second of two errors pointed out in [**A1,** 13]). This mismatch was corrected in the translation.

In the original English version of Kelley [**1**], Kuratowski's *Une méthode d'élimination des nombres transfinis des raisonnements mathématiques,* Fund. Math., **3**(1922), 76–108 appears in the bibliography, but *Sur l'opération $\overline{A}$ de l'Analysis Situs* does not. Despite there being no mention of the 14-set result in the former article, the author of the present paper gave it as reference [**2**]. Since the other article was clearly meant to be cited instead, this is what appears in the translation.

Finally, we note that the appendix beginning on the next page did not appear in the original.

Appendix

We find the Kuratowski monoids of all topological spaces X with $|X| \leq 4$. Let i and k denote the topological interior and closure operators, respectively.

As shown in **[A1]**, the Kuratowski monoid of a space is determined by the presence (or absence) of each of the following identities: (M1) $iki = ki$, (M2) $iki = ik$, (M3) $iki = i$.

Using the definitions in **[A1]**, the following table displays the identities that give rise to the six different types of space that can occur based on their Kuratowski monoid:

identities	space type
(none)	Kuratowski
(M1)	extremally disconnected (ED)
(M2)	open unresolvable (OU)
(M1), (M2)	ED and OU
(M3) ($\Rightarrow$(M1))	partition
(M2), (M3) ($\Rightarrow$(M1))	discrete

Note that every indiscrete space X with $|X| \geq 2$ is a partition space since identity (M3) clearly holds and we have $iA = \varnothing$ and $kA = X$, hence $ikiA = \varnothing$ and $ikA = X$, for every nonempty proper subset A of X.

Clearly the unique 1-point topological space is discrete. It is similarly obvious that the discrete and indiscrete 2-point spaces are discrete and partition spaces, respectively. The unique nontrivial 2-point space $X = \{a, b\}$ with topology $\{\varnothing, \{a\}, X\}$ is ED and OU since identities (M1) and (M2) hold and we have $iki(\{a\}) = X$ and $i(\{a\}) = \{a\}$.

The early appearance of an ED and OU space in this discussion hints at things to come. The frequency distribution of the various Kuratowski monoids occurring among the nine distinct (up to homeomorphism) 3-point spaces is given in the following table:

Kuratowski	ED	OU	ED and OU	partition	discrete
0	1	1	4	2	1

The frequency distribution among the 33 distinct 4-point spaces looks like this:

Kuratowski	ED	OU	ED and OU	partition	discrete
1	6	7	14	4	1

The next table lists each of the nine 3-point spaces $X = \{a, b, c\}$ and their types, where the empty set and whole space implicitly belong to each base:

#	base	space type	notes
1	$\{a\}, \{b\}, \{c\}$	discrete	
2	$\{a\}, \{b\}$	OU	$iki(\{a\}) = \{a\}$, $ki(\{a\}) = \{a, c\}$
3	$\{a\}, \{b\}, \{b, c\}$	ED and OU	$iki(\{b\}) = \{b, c\}$, $i(\{b\}) = \{b\}$
4	$\{a\}$	ED and OU	$iki(\{a\}) = X$, $i(\{a\}) = \{a\}$
5	$\{a\}, \{a, b\}$	ED and OU	$iki(\{a\}) = X$, $i(\{a\}) = \{a\}$
6	$\{a\}, \{b, c\}$	partition	
7	$\{a\}, \{a, b\}, \{a, c\}$	ED and OU	$iki(\{a\}) = X$, $i(\{a\}) = \{a\}$
8	indiscrete	partition	
9	$\{a, b\}$	ED	$iki(\{a\}) = \varnothing$, $ik(\{a\}) = X$ $iki(\{a, b\}) = X$, $i(\{a, b\}) = \{a, b\}$

The following table lists the same for the 4-point spaces $X = \{a, b, c, d\}$:

#	base	space type	notes
1	$\{a\}, \{b\}, \{c\}, \{d\}$	discrete	
2	$\{a\}, \{b\}, \{c\}$	OU	$iki(\{a\}) = \{a\}$, $ki(\{a\}) = \{a, d\}$
3	$\{a\}, \{b\}, \{c\}, \{a, b, d\}$	OU	$iki(\{a\}) = \{a\}$, $ki(\{a\}) = \{a, d\}$
4	$\{a\}, \{b\}, \{c\}, \{a, d\}$	ED and OU	$iki(\{a\}) = \{a, d\}$, $i(\{a\}) = \{a\}$
5	$\{a\}, \{b\}$	OU	$iki(\{a\}) = \{a\}$, $ki(\{a\}) = \{a, c, d\}$
6	$\{a\}, \{b\}, \{a, b, c\}$	OU	$iki(\{a\}) = \{a\}$, $ki(\{a\}) = \{a, c, d\}$
7	$\{a\}, \{b\}, \{a, c, d\}$	ED and OU	$iki(\{a\}) = \{a, c, d\}$, $i(\{a\}) = \{a\}$
8	$\{a\}, \{b\}, \{a, b, c\}, \{a, b, d\}$	OU	$iki(\{a\}) = \{a\}$, $ki(\{a\}) = \{a, c, d\}$
9	$\{a\}, \{b\}, \{a, c\}$	OU	$iki(\{a\}) = \{a, c\}$, $ki(\{a\}) = \{a, c, d\}$
10	$\{a\}, \{b\}, \{a, c\}, \{a, b, d\}$	OU	$iki(\{a\}) = \{a, c\}$, $ki(\{a\}) = \{a, c, d\}$
11	$\{a\}, \{b\}, \{a, c\}, \{a, c, d\}$	ED and OU	$iki(\{a\}) = \{a, c, d\}$, $i(\{a\}) = \{a\}$
12	$\{a\}, \{b\}, \{c, d\}$	partition	
13	$\{a\}, \{b\}, \{a, c\}, \{b, d\}$	ED and OU	$iki(\{a\}) = \{a, c\}$, $i(\{a\}) = \{a\}$
14	$\{a\}, \{b\}, \{a, c\}, \{a, d\}$	ED and OU	$iki(\{a\}) = \{a, c, d\}$, $i(\{a\}) = \{a\}$
15	$\{a\}$	ED and OU	$iki(\{a\}) = X$, $i(\{a\}) = \{a\}$
16	$\{a\}, \{a, b, c\}$	ED and OU	$iki(\{a\}) = X$, $i(\{a\}) = \{a\}$
17	$\{a\}, \{b, c, d\}$	partition	
18	$\{a\}, \{a, b\}$	ED and OU	$iki(\{a\}) = X$, $i(\{a\}) = \{a\}$
19	$\{a\}, \{a, b\}, \{a, b, c\}$	ED and OU	$iki(\{a\}) = X$, $i(\{a\}) = \{a\}$
20	$\{a\}, \{b, c\}$	Kuratowski	$iki(\{a\}) = \{a\},\ ki(\{a\}) = \{a, d\}$ $iki(\{b\}) = \varnothing,\ ik(\{b\}) = \{b, c\}$ $iki(\{a, b, c\}) = X,\ i(\{a, b, c\}) = \{a, b, c\}$
21	$\{a\}, \{a, b\}, \{a, c, d\}$	ED and OU	$iki(\{a\}) = X$, $i(\{a\}) = \{a\}$
22	$\{a\}, \{a, b\}, \{a, b, c\}, \{a, b, d\}$	ED and OU	$iki(\{a\}) = X$, $i(\{a\}) = \{a\}$
23	$\{a\}, \{b, c\}, \{b, c, d\}$	ED	$iki(\{b\}) = \varnothing,\ ik(\{b\}) = \{b, c, d\}$ $iki(\{b, c\}) = \{b, c, d\},\ i(\{b, c\}) = \{b, c\}$
24	$\{a\}, \{a, b\}, \{a, c\}$	ED and OU	$iki(\{a\}) = X$, $i(\{a\}) = \{a\}$
25	$\{a\}, \{a, b\}, \{c, d\}$	ED	$iki(\{c\}) = \varnothing,\ ik(\{c\}) = \{c, d\}$ $iki(\{a\}) = \{a, b\},\ i(\{a\}) = \{a\}$
26	$\{a\}, \{a, b\}, \{a, c\}, \{a, b, d\}$	ED and OU	$iki(\{a\}) = X$, $i(\{a\}) = \{a\}$
27	$\{a\}, \{a, b\}, \{a, c\}, \{a, d\}$	ED and OU	$iki(\{a\}) = X$, $i(\{a\}) = \{a\}$
28	indiscrete	partition	
29	$\{a, b, c\}$	ED	$iki(\{a\}) = \varnothing,\ ik(\{a\}) = X$ $iki(\{a, b, c\}) = X,\ i(\{a, b, c\}) = \{a, b, c\}$
30	$\{a, b\}$	ED	$iki(\{a\}) = \varnothing,\ ik(\{a\}) = X$ $iki(\{a, b\}) = X,\ i(\{a, b\}) = \{a, b\}$
31	$\{a, b\}, \{a, b, c\}$	ED	$iki(\{a\}) = \varnothing,\ ik(\{a\}) = X$ $iki(\{a, b\}) = X,\ i(\{a, b\}) = \{a, b\}$
32	$\{a, b\}, \{a, b, c\}, \{a, b, d\}$	ED	$iki(\{a\}) = \varnothing,\ ik(\{a\}) = X$ $iki(\{a, b\}) = X,\ i(\{a, b\}) = \{a, b\}$
33	$\{a, b\}, \{c, d\}$	partition	

Reference

[**A1**] B. J. Gardner and M. Jackson, *The Kuratowski closure-complement theorem,* New Zealand J. Math, **38**(2008), 9–44.

Mark Bowron
Las Vegas, NV USA
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