On the Operation $\overline{A}$ in Analysis Situs

by Kazimierz Kuratowski

Author's note. This paper is the first part — slightly modified — of my thesis presented May 12, 1920 at the University of Warsaw for the degree of Doctor of Philosophy.

Note. This English translation of Kuratowski's paper *Sur l'opération* $\overline{A}$ *de l'analysis situs* (in French), Fund. Math., **3**(1922), 182–199, was prepared by Mark Bowron in August 2010.

Let $\mathbb{E}^n$ denote n-dimensional Euclidean space. For $A \subset \mathbb{E}^n$, let A' denote the complement of A in $\mathbb{E}^n$, namely $A' = \mathbb{E}^n \setminus A$.

Let $\overline{A}$ denote the union of A and its limit points. We shall often use — for typographical reasons — the notation A^- instead of $\overline{A}$.

It is easy to show that each of the following hold:

$$\text{I.} \qquad \overline{A \cup B} = \overline{A} \cup \overline{B}$$
$$\text{II.} \qquad A \subset \overline{A}$$
$$\text{III.} \qquad \overline{\varnothing} = \varnothing$$
$$\text{IV.} \qquad \overline{\overline{A}} = \overline{A}.$$

In this paper we analyze these properties and their consequences. We proceed in an axiomatic fashion by assuming as given both an arbitrary set X, and an operation $\overline{A}$, such that for every subset $A \subset X$, there exists a subset $\overline{A} \subset X$ that satisfies Axioms I–IV. Furthermore, for all sets X, we shall use only those properties of sets which follow from the axioms of Couturat [**1**].

Thus, Axioms I–IV added to those of [**1**] form the foundation upon which all arguments of this paper are based (except those in small print).

If we compare the system of Axioms I–III to that used to define the abstract $\mathcal{L}$-classes of Fréchet [**2**], we recognize without difficulty that the former is more general than the latter: any $\mathcal{L}$-class satisfies Axioms I–III, but there are spaces satisfying Axioms I–IV that are not $\mathcal{L}$-classes.

§ 1. General Properties of the Closure Operation

We now establish six fundamental properties of the closure operation.

Theorem 1. $A \subset B$ implies $\overline{A} \subset \overline{B}$.

Theorem 2. $\overline{A \cap B} \subset \overline{A} \cap \overline{B}$.

Theorem 3. ([**6,** 222]) $\overline{A} \setminus \overline{B} \subset \overline{A \setminus B}$.

Theorem 4. $\overline{X} = X$.

Theorem 5. $A^{-\prime} \subset A^{\prime -}$.

Theorem 6. $A^{-\prime-\prime-\prime-} = A^{-\prime-}$.

The first three theorems follow from Axiom I. Indeed, the inclusion $A \subset B$ implies $B = A \cup B$, hence $\overline{B} = \overline{A \cup B}$, and by I, $\overline{B} = \overline{A} \cup \overline{B}$. Thus $\overline{A} \subset \overline{B}$. To establish Theorem 2, note that

$$A \cap B \subset A \text{ and } A \cap B \subset B,$$

which by Theorem 1 implies that

$$\overline{A \cap B} \subset \overline{A} \text{ and } \overline{A \cap B} \subset \overline{B}.$$

Thus

$$\overline{A \cap B} \subset \overline{A} \cap \overline{B}.$$

Finally, the obvious formula

$$A \subset A \cup B = (A \setminus B) \cup B$$

leads to

$$\overline{A} \subset \overline{(A \setminus B) \cup B} = \overline{A \setminus B} \cup \overline{B},$$

which implies Theorem 3.

Theorem 4 follows from Axiom II. Indeed, from the definition of the closure operation we have $\overline{X} \subset X$, and from II, $X \subset \overline{X}$. Thus $X = \overline{X}$.

Applying this identity, we deduce Theorem 5 from Theorem 3:

$$A^{-\prime} = X \setminus \overline{A} = \overline{X} \setminus \overline{A} \subset \overline{X \setminus A} = A^{\prime -}.$$

To establish Theorem 6, we must invoke Axioms I, II, and IV. By Axiom IV and the principle of double negation ($A'' = A$), we have

$$A^{-\prime -} = A^{-\prime --} = A^{-\prime --\prime\prime}.$$

By Theorem 5,

$$A^{-\prime --\prime} \subset A^{-\prime -\prime -},$$

hence

$$A^{-\prime -\prime -\prime} \subset A^{-\prime --\prime\prime} = A^{-\prime -}.$$

Thus, by Theorem 1 and Axiom IV,

(1) $$A^{-\prime -\prime -\prime -} \subset A^{-\prime -}.$$

On the other hand,

$$A^{-\prime -\prime} \subset A^{-\prime\prime -} = A^{--} = A^{-}.$$

Thus

$$(A^{-\prime -\prime})^{-\prime -} \supset (A^{-})^{-\prime -} = A^{-\prime -},$$

that is,

(2) $$A^{-\prime -\prime -\prime -} \supset A^{-\prime -}.$$

Formulas (1) and (2) give Theorem 6.

We shall return to this theorem in § 4.

With regard to unions and intersections of infinitely many sets, we have the following theorem:

Theorem 2a. Letting $\{A_i\}$ denote an arbitrary family of sets indexed by the variable i, we have

$$\overline{\bigcap_i A_i} \subset \bigcap_i \overline{A_i} \quad \text{and} \quad \bigcup_i \overline{A_i} \subset \overline{\bigcup_i A_i}.$$

Proof. For every index k we have

$$\bigcap_i A_i \subset A_k \quad \text{hence} \quad \overline{\bigcap_i A_i} \subset \overline{A_k},$$

thus

$$\overline{\bigcap_i A_i} \subset \bigcap_k \overline{A_k} = \bigcap_i \overline{A_i}.$$

Similarly, the inclusion $A_k \subset \bigcup_i A_i$ implies

$$\bigcup_i \overline{A_i} = \bigcup_k \overline{A_k} \subset \overline{\bigcup_i A_i}.$$ Q.E.D.

We now generalize Theorems 1–3. Let σ denote any finite sequence composed of the symbols " $-$ " and " $'$ ". We say that σ is *even* if the symbol " $'$ " occurs an even number (≥ 0) of times in σ. Otherwise we say σ is *odd*.

Based on Theorem 1 and the principle of contraposition (by which $A \subset B$ implies $B' \subset A'$), one easily establishes the following formulas:

Case 1. When σ is even,

$$A \subset B \ \text{ implies } \ A^\sigma \subset B^\sigma, \tag{3}$$

$$(A \cap B)^\sigma \ \subset \ A^\sigma \cap B^\sigma \ \subset \ A^\sigma \cup B^\sigma \ \subset \ (A \cup B)^\sigma, \tag{4}$$

$$\Big(\bigcap_i A_i\Big)^\sigma \ \subset \ \bigcap_i A_i^\sigma \ \subset \ \bigcup_i A_i^\sigma \ \subset \ \Big(\bigcup_i A_i\Big)^\sigma. \tag{5}$$

Case 2. When σ is odd,

$$A \subset B \text{ implies } B^\sigma \subset A^\sigma, \tag{6}$$

$$(A \cup B)^\sigma \subset A^\sigma \cap B^\sigma \subset A^\sigma \cup B^\sigma \subset (A \cap B)^\sigma, \tag{7}$$

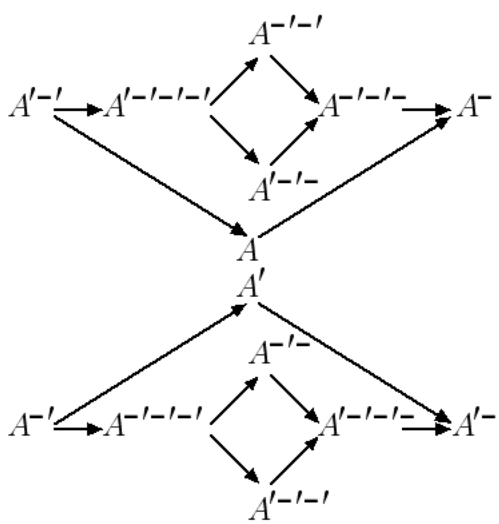
$$\Big(\bigcup_i A_i\Big)^\sigma \ \subset \ \bigcap_i A_i^\sigma \ \subset \ \bigcup_i A_i^\sigma \ \subset \ \Big(\bigcap_i A_i\Big)^\sigma. \tag{8}$$

To generalize Theorem 3, we show that

$$\begin{cases} A^\sigma \backslash B^\sigma \ \subset \ \overline{A \backslash B} \ \ (\sigma \text{ even}) \\ A^\sigma \backslash B^\sigma \ \subset \ \overline{B \backslash A} \ \ (\sigma \text{ odd}). \end{cases} \tag{9}$$

Indeed, by Theorem 3, $\overline{A} \backslash \overline{B} \subset \overline{A \backslash B}$; at the same time, since $A' \backslash B' = B \backslash A$, we have $A' \backslash B' \subset \overline{B \backslash A}$. If (9) holds for a sequence τ and $A^\tau \backslash B^\tau \subset \overline{A \backslash B}$, then $A^{\tau -} \backslash B^{\tau -} \subset \overline{\overline{A \backslash B}} = \overline{A \backslash B}$ and $A^{\tau\prime} \backslash B^{\tau\prime} = B^\tau \backslash A^\tau \subset \overline{B \backslash A}$. Similarly, if $A^\tau \backslash B^\tau \subset \overline{B \backslash A}$, then $A^{\tau -} \backslash B^{\tau -} \subset \overline{B \backslash A}$ and $A^{\tau\prime} \backslash B^{\tau\prime} \subset \overline{A \backslash B}$. We conclude that (9) holds for all σ by induction.

Based on the theorems proved so far, we can construct the following table (T) of relations, which will be useful in what follows:



The presence of $A^\sigma \to A^\tau$ in (T) means $A^\sigma \subset A^\tau$ for all sets A.

§ 2. Fundamental Concepts of Topology

In this section we define some topological concepts that depend on the closure operation.

A set A is said to be *closed*, when $A = \overline{A}$.

By Axiom I, the union of two closed sets is closed. By Axiom II and Theorem 2a, the intersection of any collection of closed sets is closed.

When A is closed, it is easy to see that table (T) reduces to:

$$A'^{-\prime} \to A'^{-\prime-} \to A,$$
$$A' \to A'^{-\prime-\prime} \to A'^{-}.$$

A set A is said to be *connected* if A is not a union of two disjoint nonempty closed sets (we say that a connected set A is *connected in the strict sense*, if A contains more than one point).

It can be shown, using Axioms I and II, that:

Theorem 1^0. The union of two non-disjoint connected sets is connected (see [**7,** 212], Theorem IV$'$; in the same note several general properties of connected sets are proved using Axioms I and II).

Theorem 2^0. If the union and intersection of two closed sets A and B are each connected, then A and B are also each connected (see [**6,** 211], Theorem I).

For every set A, we have the following decomposition:

$$A = (A \cap A'^{-}) \cup (A \cap A'^{-\prime}) = (A \cap A'^{-}) \cup A'^{-\prime}.$$

We call $A \cap A'^{-}$ the *border* of A (Hausdorff uses this same term in [**3,** 214]). We call $A'^{-\prime}$ the *interior* of A, or the set of interior points of A.

We consider two cases, based on whether the interior or border is empty.

Case 1. Let $A'^{-\prime} = \varnothing$, so $A = A \cap A'^{-}$. In this case we call A a *boundary set*.

It follows from formula (3) that any subset of a boundary set is also a boundary set.

Theorem 7. The border of any set is a boundary set.

Proof. By (4),

$$(A \cap A'^{-})'^{-\prime} \subset A'^{-\prime} \cap A'^{-\prime-\prime}.$$

By table (T),

$$A'^{-\prime-\prime} \subset A'^{-}.$$

Therefore

$$(A \cap A'^{-})'^{-\prime} \subset A'^{-\prime} \cap A'^{-} = \varnothing. \qquad \text{Q.E.D.}$$

A set A is said to be *nowhere dense* when $\overline{A}$ is a boundary set; that is, when $A^{-\prime-\prime} = \varnothing$.

It follows that any subset of a nowhere dense set is nowhere dense. Every nowhere dense set is clearly a boundary set. On the other hand, every closed boundary set is nowhere dense.

As the border of a closed set is an intersection of two closed sets, hence closed, we deduce from Theorem 7 the following corollary:

Corollary. The border of a closed set is nowhere dense.

Theorem 8. The union of two nowhere dense sets is nowhere dense.

Proof. Let A and B be two nowhere dense sets. This is equivalent to $A^{-\prime-} = X$ and $B^{-\prime-} = X$. We need to show that $(A \cup B)^{-\prime-} = X$.

We have

$$(A \cup B)^{-\prime-} = (\overline{A} \cup \overline{B})'^{-} = (A^{-\prime} \setminus B^{-})^{-}.$$

By Theorem 3,

$$(A^{-\prime}\setminus B^{-})^{-}\supset A^{-\prime-}\setminus B^{-}=X\setminus B^{-}=B^{-\prime}.$$

Thus

$$(A\cup B)^{-\prime-}\supset B^{-\prime}.$$

Therefore, by Axiom IV and Theorem 1,

$$(A\cup B)^{-\prime-}=(A\cup B)^{-\prime--}\supset B^{-\prime-}=X. \qquad \text{Q.E.D.}$$

Case 2. Let $A\cap A^{\prime-}=\varnothing$, so $A=A^{\prime-\prime}$. In this case we say that A is *open*.

It follows immediately that a set A is open if and only if its complement $A^{\prime}$ is closed. Hence, the union of any collection of open sets is open and the intersection of two open sets is open.

Note that the interior of every set is open, for $(A^{\prime-\prime})^{\prime-\prime}=A^{\prime--\prime}=A^{\prime-\prime}$. The interior of A is also the largest open set contained in A. Indeed, if $E\subset A$, by (3) we have $E^{\prime-\prime}\subset A^{\prime-\prime}$; thus if E is an open set, we have $E=E^{\prime-\prime}\subset A^{\prime-\prime}$.

Using the closure operation we can also define the notion of boundary: we call the intersection $\overline{A}\cap A^{\prime-}$ the *boundary* of A.

Consider the obvious decomposition:

$$\overline{A}\cap A^{\prime-}\;=\;(\overline{A}\cap A^{\prime-}\cap A)\;\cup\;(\overline{A}\cap A^{\prime-}\cap A^{\prime}).$$

Since $A\subset\overline{A}$ and $A^{\prime}\subset A^{\prime-}$, it follows that

$$\overline{A}\cap A^{\prime-}\;=\;(A\cap A^{\prime-})\;\cup\;(\overline{A}\cap A^{\prime});$$

that is, the boundary of A is the union of the border of A and the border of $A^{\prime}$.

When A is closed, its boundary is reduced to $A\cap A^{\prime-}$; when A is open, it is $\overline{A}\cap A^{\prime}$.

Janiszewski [**5**] has established several other properties of the boundary using Axioms I–IV.

The concept of border leads in a natural way to that of *residue*.

The set $\overline{A}\setminus A$ is the border of the complement of A. Consider the border of the complement of $\overline{A}\setminus A$; this is the set $\overline{\overline{A}\setminus A}\setminus(\overline{A}\setminus A)=(A\cap\overline{\overline{A}\setminus A})\cup(\overline{\overline{A}\setminus A}\setminus\overline{A})=A\cap\overline{\overline{A}\setminus A}$, since $\overline{\overline{A}\setminus A}\setminus\overline{A}=\varnothing$.

Hausdorff [**3,** 281] calls $A\cap\overline{\overline{A}\setminus A}$ the *residue* of A. We denote this set by A_r.

To better illustrate the topological meaning of this concept, we introduce the term *locally closed*.

We say that E is a *neighborhood* of a point p, when p is situated in the interior of E. In a similar way, a subset E of A is said to be a *relative neighborhood* of p with respect to A, when p is situated in the relative interior of E; that is, when $p\in E\setminus\overline{A\setminus E}$.

We say that a set A is *locally closed* at a point p, if there exists a relative neighborhood of p with respect to A that is closed and bounded.

We show that A_r consists of all points in A where A is not locally closed.

Suppose $p\notin A_r$. Then $p\in A\setminus\overline{\overline{A}\setminus A}$, hence there exists a sphere S with center p such that $S\cap(\overline{A}\setminus A)=\varnothing$. The closed and bounded set $S\cap A$ is indeed a relative neighborhood of p with respect to A. Thus A is locally closed at p.

On the other hand, suppose E is a relative neighborhood of p with respect to A. We can then surround the point p with a sphere S such that $S\cap A\subset E$, which leads to $S\cap(\overline{A}\setminus E)=\varnothing$. It follows that p is neither a limit point of $\overline{A}\setminus E$ nor, as a result, a limit point of $\overline{A}\setminus A$. In other words, $p\notin A\cap\overline{\overline{A}\setminus A}$. Q.E.D.

Since the concepts of relative neighborhood and bounded and closed set are topological invariants, the same is true of the property of a set A being locally closed at a point p. We consider some consequences of the invariance of this property.

Suppose $A_r = \varnothing$. This implies A is locally closed everywhere. The equality $A_r = \varnothing$ is therefore invariant. Note, this equality is equivalent to the assumption that A is a difference of two closed sets [**3,** 281]. Thus, the property of being a difference of two closed sets is invariant [**8**].

The equality $A_r = A$ is also invariant. As shown by Hausdorff, this never occurs for sets (nonempty) which are both F_σ and G_δ. Suppose A is a *homogeneous* set; that is, for every pair of points p and q in A, there exists a homeomorphism of A into itself that sends p to q. This implies $A_r = \varnothing$ or $A_r = A$. Hence if A is both F_σ and G_δ, then $A_r = \varnothing$. Thus A is a difference of two closed sets. We conclude that a homogeneous set which is both F_σ and G_δ is a difference of two closed sets.

The concept of locally closed set also merits attention with regard to Zermelo's axiom of choice, as opposed to *actual* choice.

It is known that associated with each closed and bounded set is one of its elements. Indeed, let $x_1, x_2, \dots, x_n$ represent the coordinates of the space under consideration. If A is a closed and bounded set, then there exist points of A with minimal first coordinate x_1. If only one such point exists, we associate it with A. Otherwise, consider among the points with minimal first coordinate x_1, those with minimal second coordinate x_2, and so on. In every case we eventually arrive at a definite point $f(A)$ associated with A.

One can also define the function $f(A)$ when A contains interior points or isolated points. The concept of locally closed set permits us to generalize these results: we can define such a point $g(A)$ of A whenever A satisfies $A \neq A_r$.

Let the sequence $\{S_1, S_2, S_3, \dots\}$ denote the set of spheres with rational centers and radii in some prescribed order. If A is locally closed at one of its points, then there exist spheres S_i whose intersections with A are each nonempty, closed, and bounded. Let $n(A)$ denote the smallest index among these spheres.

The set $S_{n(A)} \cap A$ being closed and bounded, define

$$g(A) = f(S_{n(A)} \cap A).$$

Therefore, for every set A such that $A \neq A_r$, it is possible to associate with A a definite point of A without using the axiom of choice.

§ 3. Relativization

Let $R \subset X$. We deduce from Axioms I–IV the following formulas:

$$R \cap \overline{A \cup B} = (R \cap \overline{A}) \cup (R \cap \overline{B}),$$
$$A \subset R \cap \overline{A},$$
$$R \cap \overline{R \cap \overline{A}} = R \cap \overline{A},$$

where A and B denote arbitrary subsets of R.

These formulas show that the operation $R \cap \overline{A}$ satisfies Axioms I–IV for every $A \subset R$. Therefore, if we define $A^R = R \setminus A$, we can replace A' with A^R and $\overline{A}$ with $R \cap \overline{A}$ in all theorems deduced from Axioms I–IV. We say that we have *relativized* these theorems. The same process leads to *relative concepts* (as Hausdorff calls them — see [**3,** 240]).

Assuming, in particular, the set R is closed, one relativizes theorems by only replacing the symbol "$'$" with the symbol "R", since $R \cap \overline{A} = \overline{A}$.

By relativizing the definition of closed set, we arrive at the following definition: a subset A of R is *closed in* R, when $A = R \cap \overline{A}$. Thus, a set that is closed in R is the intersection of R and a closed set. The converse is also true: the intersection of R and a closed set is always closed in R.

Similarly, a set A is *open in* R when $A^R = R \cap A^{R-}$. Note, a set is open in R if and only if it is the intersection of an open set and R.

We arrive at an important concept by relativizing nowhere dense sets with respect to connected sets. We say that a connected set K is a *continuum of condensation* with respect to a connected set C, if K is nowhere dense in C (see [**4**]). Relativizing Theorems 8 and 1^0 from § 2, we conclude that the union of two non-disjoint continua of condensation is also a continuum of condensation ([**4**], Theorem VI).

Let σ be a finite sequence consisting of "$-$" and "$\prime$". We denote by σ_R the sequence obtained from σ by replacing "$\prime$" with "R". Assuming the set R is closed, we establish the following formulas:

$$\text{(10)}\qquad \begin{cases} A^{\sigma} \subset A^{\sigma_R} & (\sigma \text{ even}) \\ A^{\sigma_R} \subset A^{\sigma} & (\sigma \text{ odd}). \end{cases}$$

Proof. Note first that for every σ we have $A^{\sigma_R} \subset R$, since $A^- \subset R$ and $A^R \subset R$.

With formula (10) being true for σ consisting of a single element, suppose it holds for $\sigma = \tau$.

Case 1. When τ is even, we have $A^{\tau} \subset A^{\tau_R}$. Thus $A^{\tau -} \subset A^{\tau_R -}$ and $A^{\tau_R \prime} \subset A^{\tau \prime}$. Since $A^{\tau_R R} \subset A^{\tau_R \prime}$, this implies $A^{\tau_R R} \subset A^{\tau \prime}$.

Case 2. When τ is odd, we have $A^{\tau_R} \subset A^{\tau}$. Thus $A^{\tau_R -} \subset A^{\tau -}$ and $A^{\tau \prime} \subset A^{\tau_R \prime}$. But $\tau\prime$ being even, it follows from table (T) that $A^{\tau \prime} \subset \overline{A} \subset R$. Hence

$$A^{\tau \prime} \subset R \cap A^{\tau_R \prime} = A^{\tau_R R}.$$

Formulas (10) therefore hold by induction for all σ. Q.E.D.

§ 4. Regular Closed Sets

We call a set A *regular closed*, when

$$\text{(11)}\qquad A^{\prime - \prime -} = A.$$

For A to be regular closed, it is necessary and sufficient that there exist an open set D such that $A = \overline{D}$ (with the additional assumption that D is bounded, Lebesgue calls $\overline{D}$ a *closed domain* in [**9,** 273]).

Indeed, if $A = A^{\prime - \prime -}$, then $A = (A^{\prime - \prime})^-$ where the set $A^{\prime - \prime}$ is open. On the other hand, if D is open and $A = \overline{D}$, it follows from the definition of an open set that $A = (D^{\prime - \prime})^-$. But $A^{\prime - \prime -} = D^{\prime - \prime - \prime - \prime -} = D^{\prime - \prime -}$ by Theorem 6, so $A^{\prime - \prime -} = A$.

We call the intersection $A \cap A^{\prime - \prime -}$ the *regular part* of A. The interior of A is obviously contained in the regular part of A. Thus the difference between A and its regular part, being a subset of the border of A, is a boundary set (Theorem 7); in addition, when A is closed, it is nowhere dense. In this case $A \cap A^{\prime - \prime -} = A^{\prime - \prime -}$.

Examples. In the plane a closed disk is a regular closed set. Two disjoint closed disks united by a line segment form a non-regular set: the regular part consists of the two disks.

Let R be the set of points (x, y) subject to the conditions:

$$\begin{cases} \text{when} & -1 \leq x < 0, & y = 0, \\ \text{when} & x = 0, & -1 \leq y \leq 1, \\ \text{when} & 0 < x \leq 1, & y = \sin \frac{1}{x}. \end{cases}$$

The subset with $x \leq 0$ is not regular in R. The subset with $x \geq 0$ is regular in R.

We prove some properties of regular closed sets.

The topological meaning of Theorem 6 can be expressed this way:

Theorem 9. If A is closed, $A^{\prime -}$ is regular closed.

Theorem 10. The union of two regular closed sets is regular closed; more generally, if the sets A_i are each regular closed, then so is the set $\overline{\cup_i A_i}$.

Proof. Note that when A is closed, the inclusion $A \subset A^{\prime - \prime -}$ implies, by the reduced table (T), that A is regular closed.

Suppose $A = A'^{-'-}$ and $B = B'^{-'-}$. By (4),

$$A \cup B \;=\; A'^{-'-} \cup B'^{-'-} \;\subset\; (A \cup B)'^{-'-}.$$

Thus $A \cup B$ is regular closed.

Similarly, if $A_i = {A_i}'^{-'-}$, we have by (5),

$$\bigcup_i A_i \;=\; \bigcup_i {A_i}'^{-'-} \;\subset\; (\bigcup_i A_i)'^{-'-}.$$

Hence, by (3),

$$(\bigcup_i A_i)'^{-'-} \;\subset\; (\bigcup_i A_i)^{-'-'-}.$$

We conclude that

$$\overline{\bigcup_i A_i} \;\subset\; (\bigcup_i A_i)'^{-'--} \;=\; (\bigcup_i A_i)'^{-'-} \;\subset\; (\bigcup_i A_i)^{-'-'-}. \qquad \text{Q.E.D.}$$

Theorem 11. If the set R is regular closed and A is a closed subset of R, then the regular part of A equals the regular part of A with respect to R.

Proof. By (10), $A'^{-'-} \subset A^{R-R-}$. It remains to show that $A^{R-R-} \subset A'^{-'-}$.

By (9) we have

$$R'^{-'-} \setminus A'^{-'-} \;\subset\; \overline{R \setminus A}.$$

Since $R \;=\; R'^{-'-}$, this implies

$$R \setminus A'^{-'-} \;\subset\; A^{R-}.$$

Since the inclusions $X \setminus Y \subset Z$ and $X \setminus Z \subset Y$ are equivalent, we get

$$R \setminus A^{R-} \;\subset\; A'^{-'-}.$$

Thus

$$A^{R-R} \;\subset\; A'^{-'-}.$$

Therefore, by Axiom IV and Theorem 1,

$$A^{R-R-} \;\subset\; A'^{-'-}. \qquad \text{Q.E.D.}$$

Corollary 1. In order for a subset A of a regular closed set R to be regular closed, it is necessary and sufficient for A to be regular closed with respect to R.

Relativizing this corollary with respect to a closed set S, we see that if A is regular closed with respect to R and R is regular closed with respect to S, then A is regular closed with respect to S. Put another way:

Corollary 2. The property of being a regular closed set with respect to a closed set is transitive.

Corollary 3. If R is regular closed and A is closed, then $\overline{R \setminus A}$ is regular closed.

Proof. We have

$$R \setminus A \;=\; R \setminus (R \cap A) \;=\; (R \cap A)^R,$$

hence

$$\overline{R \setminus A} \;=\; (R \cap A)^{R-}.$$

Relativizing Theorem 9 with respect to R allows us to conclude, since $R \cap A$ is a closed subset of R, that $(R \cap A)^{R-}$ is regular closed with respect to R. Thus, by Corollary 1, it is regular closed. Q.E.D.

Note that if the regular part of a closed set A is empty ($A'^{-'-} = \varnothing$), then A is nowhere dense. By Axiom III the converse is also true: the regular part of a nowhere dense set is empty. We deduce the following corollary of Theorem 11:

Corollary 4. In order for a closed subset of a regular closed set R to be nowhere dense, it is necessary and sufficient for it to be nowhere dense with respect to R.

In addition to regular closed sets, we shall consider regular open sets. We say that A is *regular open*, when

$$A = A^{-\prime-\prime}.$$

Based on the identity $A'' = A$, it is easy to see that the complement of a regular closed set is regular open, and vice versa. Thus many properties of regular open sets follow directly from theorems on regular closed sets. In particular, when A is open, $A^{-\prime}$ is regular open.

The following is an important property of regular open sets that is independent of Axioms I–IV.

We know that an open set is a union of disjoint connected open sets. The boundary of each of these components is contained in the boundary of the whole. We show that if C is a component of an regular open set D, then C is also regular open.

To establish the equality $C = C^{-\prime-\prime}$, it suffices to show, given table (T), that $C^{-\prime-\prime} \subset C$. Note that

$$(C^{-\prime-\prime} \cap C') \subset (C^{-} \cap C') \subset (D^{-} \cap D').$$

By (3),

$$C^{-\prime-\prime} \subset D^{-\prime-\prime} = D.$$

Therefore,

$$(C^{-\prime-\prime} \cap C') \subset (D^{-} \cap D' \cap D) = \varnothing.$$

This proves that $C^{-\prime-\prime} \subset C$.

Aside. The boundaries of regular open sets have some interesting properties. Lebesgue established some of them in the previous volume of this journal. He has used the term $(n-1)$-*dimensional boundary* when the regular open set is bounded and connected, where n is the number of dimensions in the space under consideration.

§ 5. Logical Analysis of Axioms I–IV

Table (T) contains 14 sets that can be obtained from a given set A by combining the closure and complement operations. Based on Axiom IV, Theorem 6, and the principle of double negation ($A'' = A$), we easily show that this number of sets can never exceed 14. We will also show that it cannot, in general, be reduced.

To clarify the problem, we say that a sequence σ of the symbols "$-$" and "$\prime$" is *irreducible*, if for every subsequence τ of σ there exists a set A (of points of Euclidean space) such that $A^{\tau} \neq A^{\sigma}$. Our task is to prove that every sequence appearing in table (T) is irreducible.

This will follow from the following proposition:

Proposition. Every theorem of the form "$A^{\sigma_1} \subset A^{\sigma_2}$ (for any set A of points)" is represented in table (T). (Note, in addition to those shown in the table itself, we consider to be represented in table (T) any theorems which hold by transitivity: for example $A^{-\prime-\prime} \to A^{-}$, since $A^{-\prime-\prime} \to A^{-\prime-\prime-} \to A^{-}$.)

Proof. To establish the proposition we must show that if σ_1 and σ_2 are two irreducible sequences and the inclusion $A^{\sigma_1} \to A^{\sigma_2}$ is not represented in table (T), then there exists a set A such that $A^{\sigma_1} \not\subset A^{\sigma_2}$.

Consider the set composed of all numbers in the segment $(0, 1/2)$, all rational numbers in the segment $(1/2, 1)$, and the number 2.

We easily recognize that the following inclusions do not hold for this set:

$$(Q_1)\ \ A \to A^{-\prime-\prime-}, \qquad (Q_2)\ \ A^{\prime-\prime-} \to A^{-\prime-\prime},$$

$$(Q_3)\ \ A^{-\prime-\prime} \to A^{\prime-\prime-}, \qquad (Q_4)\ \ A^{-\prime} \to A^{-}.$$

With this established, we will show that no relationship of the form $A^{\sigma_1} \to A^{\sigma_2}$ beyond those already represented in the table can hold in general.

Suppose first that the sequence σ is even.

By (Q_1), if $A \to A^\sigma$, then σ is "$-$"; moreover, $\overline{A} \to A^\sigma$ does not occur, since $A \subset \overline{A}$. On the other hand, if there existed an even σ other than "$\prime - \prime$" satisfying $A^\sigma \to A$, then we would have $A^{\prime - \prime - \prime - \prime} \to A$, hence $A' \to A^{\prime - \prime - \prime -}$, and, letting $A' = B$, $B \to B^{- \prime - \prime -}$. This being (Q_1), we conclude no such σ exists. On the other hand, it is clear that no inclusion of the form $A^\sigma \to A^{\prime - \prime}$ can occur.

Since the inclusions (Q_2) and (Q_3) do not hold, we cannot add any new inclusion to the top half of table (T). Moreover, since the lower half is obtained from the upper by the principle of contraposition, the lower half is also complete.

It remains to show that there exist no general inclusions between two sets from different halves of the table. Suppose $A^{\sigma_1} \to A^{\sigma_2}$. By symmetry we can assume σ_1 is odd and σ_2 is even. It follows immediately that $A^{-\prime} \to A^{-}$, that is, inclusion (Q_4), which is impossible. Q.E.D.

It follows, in particular, that every sequence σ in table (T) is irreducible, for otherwise the table would contain sequences σ and τ such that both $A^\sigma \to A^\tau$ and $A^\tau \to A^\sigma$.

As table (T) was deduced from Axioms I–IV, there exists in Euclidean space no theorem of the form $A^\sigma \to A^\tau$ that is independent of these axioms. Thus, if we wanted to add to the system of Axioms I–IV a new axiom, it could only be defined using the operations $\overline{A}$ and A'.

The problems addressed in this section can be seen from a broader perspective when one considers, in addition to the operations $\overline{A}$ and A', those of union and intersection. Denote by $\varphi(A)$ an arbitrary function of A obtained using these four operations. An essential question is, do there exist any theorems of the form $\varphi(A) = \varnothing$ that are independent of Axioms I–IV? For the special case when $\varphi(A) = A^{\sigma_1} \cap A^{\sigma_2}$, the answer is — as we have seen — negative. The general case will not be treated here.

We shall only prove a certain property of the function $\varphi(A)$.

We have previously shown that using the operation A^σ, one just obtains 14 different functions. However, the operation $\varphi(A)$ leads to an infinite number of functions.

For a proof of this, consider the operation $\varphi(A) = A \cap \overline{\overline{A} \setminus A}$. Let B be a totally ordered set of order type ω^ω. Let $a \in B$ and define $A = B \setminus \{a + \omega,\, a + \omega^3,\, a + \omega^5, \ldots\}$.

It is easy to see that, under the order topology, $\varphi(A)$ consists of the elements of A of the form $a + \omega^n$ with $n \geq 2$, $\varphi(\varphi(A))$ those of the form $a + \omega^n$ with $n \geq 4$, and so on. The operation $\varphi(A)$ thus leads to an infinite number of distinct sets.

Axioms I–IV are independent of each other. Each of the following four examples proves the independence of the corresponding axiom, where it is assumed that $X = \{a, b, c\}$:

I. $\overline{\varnothing} = \varnothing$, $\overline{\{a\}} = \{a\}$, $\overline{\{b\}} = \{b\}$, $\overline{\{c\}} = \{c\}$ and for all other A we set $\overline{A} = X$;
II. $\overline{A} = \varnothing$ for all $A \subset X$;
III. $\overline{A} = X$ for all $A \subset X$;
IV. $\overline{\varnothing} = \varnothing$, $\overline{\{a\}} = \{a, b\}$, $\overline{\{b\}} = \{b, c\}$, $\overline{\{c\}} = \{a, c\}$ and for all other A we set $\overline{A} = X$.

To conclude this paper we consider a different system of axioms that could also serve as the foundation for our reasoning.

If we choose as a starting point the concept of derived set (A^d), then the following axioms hold:

$$\begin{array}{ll} \text{I}'. & (A \cup B)^d = A^d \cup B^d \\ \text{II}'. & X^d = X \\ \text{III}'. & \varnothing^d = \varnothing \\ \text{IV}'. & A^{dd} \subset A^d. \end{array}$$

Aside. Some properties of this kind have been studied by F. Riesz (International Congress of Mathematicians, Rome 1908) and M. Fréchet (*Bull. Sc. Math.* 1918).

To prove the independence of these axioms, just replace the symbol "$-$" with "d" in the four examples above. By modifying Theorems 1–6 similarly, we obtain theorems that follow from Axioms I$'$–IV$'$.

Similarly, by considering an arbitrary finite sequence σ consisting of the symbols "d" and "$\prime$", it can be shown that every theorem of the form $A^{\sigma_1} \to A^{\sigma_2}$ results from Axioms I$'$–IV$'$. As for the function φ, it will, perhaps, be interesting to see that there exists for a space of $n \geq 2$ dimensions a theorem that is both independent of Axioms I$'$–IV$'$, and stated using the four operations: A^d, A', union, and intersection.

This theorem is

$$((A \cap A'^d) \cup (A' \cap A^d))^d = A^d \cap A'^d,$$

which means that the derived set of the boundary of A equals the intersection of the derived set of A and the derived set of the complement of A. (One might, indeed, replace "$=$" with "$\supset$", for the opposite inclusion can be deduced from Axioms I$'$–IV$'$.)

The independence of this theorem is shown by letting X be the Euclidean line and A any line segment.

To express $\overline{A}$ in terms of A^d, we need only consider the formula $\overline{A} = A \cup A^d$. But we cannot similarly go through any simple path from $\overline{A}$ to A^d. More precisely: there is no function $\varphi(A)$, in the sense previously established, such that $A^d = \varphi(A)$.

Indeed, suppose that such a function exists. Let X be the Euclidean line and let

$$A = \{0\} \cup \{1/n : n = 1, 2, \ldots\}.$$

We obviously have $\overline{A} = A$ and $A'^{-} = X$. Hence only four sets are obtainable from A using the function φ: A, A', $\varnothing$, and X. Note that A^d is not among these four sets.

The set A^d can therefore not be defined by any identity of the form $A^d = \varphi(A)$. However, we can define A^d in terms of the closure operation as follows: A^d is the set of points p such that $p \in \overline{A \setminus \{p\}}$.

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