Closure, Complement, and Arbitrary Union

10577 [1997, 169]. *Proposed by Mark Bowron and Stanley Rabinowitz, MathPro Press, Westford, MA.* It is well known that no more than 14 distinct sets can be obtained from one set in a topological space by repeatedly applying the operations of closure and complement in any order. Is there any bound on the number of sets that can be generated if we further allow arbitrary unions to be taken in addition to closures and complements?

Solution by John Rickard, Advanced Telecommunications Modules Ltd., Cambridge, UK. No. Let A_n be the set of reals in the half-open interval $[0, 1)$ that have a binary expansion with exactly n 1's. For example, $A_0 = \{0\}$, and $A_1 = \{1/2, 1/4, 1/8, \ldots\}$. Assuming the usual topology, the closure of A_n is the union $A_0 \cup A_1 \cup \cdots \cup A_n$. For $n \geq 0$, let $S_{2n} = \bigcup_{k=0}^{n} A_{2k}$ and $S_{2n+1} = \bigcup_{k=0}^{n} A_{2k+1}$.

The closure of S_n is $A_0 \cup A_1 \cup \cdots \cup A_n$, so closure$(S_n) - S_n = S_{n-1}$, since the A_i are mutually disjoint. Thus each of the sets $S_0, \ldots, S_n$ can be generated from S_n under the operations of closure and set difference. The operation of set difference may in turn be defined in terms of union and complement using DeMorgan's laws, so this example shows that there is no finite bound.

It is also possible to generate infinitely many sets from one set of reals. An example of such a set is $S_0 \cup (2 + S_1) \cup (4 + S_2) \cup (6 + S_3) \cup \cdots$.

Editorial comment. Luke Pebody noted that for a given positive integer n, there is a topological space and set in the space such that exactly n sets can be generated using these three operations if and only if $n = 2^k$ for some k. Both Pebody and the proposers noted that there is in fact no *transfinite* bound for the number of sets generated. Given any cardinal number $\mathfrak{m}$, there is a topological space and a set in the space such that at least $\mathfrak{m}$ different sets are generated by these three operations.

The result about 14 sets first appeared in C. Kuratowski, Sur l'operation A de l'analysis situs, *Fund. Math.* 3 (1922) 182-199. Previous MONTHLY problems related to the 14 sets problem include 5569 [1968, 199; 1971, 411], 5996 [1974, 1034; 1978, 283], and 6260 [1979, 226; 1980, 680].

Solved also by B. Burdick, E. S. Langford, L. Pebody (U. K.), and the proposers.