Dave Rusin's 14-set TEXplanation dated 27 Jun 2001, re: Math 550 HELP !!!
http://www.math.niu.edu/~rusin/known-math/94/kuratowski

What is the largest number of sets one can form from a single subset of a topological space, taking only closures and complements? This is Kuratowski's problem.

Let us write $\sigma(A)$ for the closure of A and $\tau(A)$ for the complement of A (in the larger space X). Note that σ and τ may be viewed as functions from the power set of X to itself, and viewing them as functions we may take their composites (with the associative law satisfied). As in a group theory discussion, say, we may denote composition by juxtaposition, that is, fg means "first do g, then do f."

Clearly $\sigma^2 = \sigma$ and $\tau^2 = 1$ (the identity function) so the only iterates of interest are alternating products of σ's and τ's. We use just a few other basic facts about σ and τ. First, whenever applied to any subsets B, C of X, we have

$$(1)\ B \subseteq C \longrightarrow \sigma(B) \subseteq \sigma(C)$$
$$(2)\ B \subseteq C \longrightarrow \tau(C) \subseteq \tau(B)$$

which together tell us that

$$(3)\ B \subseteq C \longrightarrow \sigma\tau(C) \subseteq \sigma\tau(B)$$

The other basic fact is that for any set B,

$$(4)\ B \subseteq \sigma(B)$$

to which we apply (3) once to get

$$(5)\ \sigma\tau\sigma(B) \subseteq \sigma\tau(B)$$

and then again to get

$$(6)\ \sigma\tau\sigma\tau(B) \subseteq \sigma\tau\sigma\tau\sigma(B).$$

Here's the trick, though: these statements apply to *any* subset B of X. Given a single subset Y of X we simply apply (6) to $B = \tau\sigma(Y)$ and (5) to $B = \tau\sigma\tau\sigma(Y)$ to conclude

$$\sigma\tau\sigma\tau\tau\sigma(Y) \subseteq \sigma\tau\sigma\tau\sigma\tau\sigma(Y) \subseteq \sigma\tau\tau\sigma\tau\sigma(Y).$$

But both ends of these inclusions equal $\sigma\tau\sigma(Y)$, so that $\sigma\tau\sigma\tau\sigma\tau\sigma(Y) = \sigma\tau\sigma(Y)$. That is, $\sigma\tau\sigma\tau\sigma\tau\sigma$ and $\sigma\tau\sigma$ have the same effect on any subset Y in the power set of X. That means the functions $\sigma\tau\sigma\tau\sigma\tau\sigma$ and $\sigma\tau\sigma$ are equal, so that the complete set (actually semigroup) of functions generated by σ and τ is

$$1, \sigma, \tau\sigma, \sigma\tau\sigma, \tau\sigma\tau\sigma, \sigma\tau\sigma\tau\sigma, \tau\sigma\tau\sigma\tau\sigma,$$
$$\tau, \sigma\tau, \tau\sigma\tau, \sigma\tau\sigma\tau, \tau\sigma\tau\sigma\tau, \sigma\tau\sigma\tau\sigma\tau, \tau\sigma\tau\sigma\tau\sigma\tau$$

since all longer strings of composites involve $\sigma\sigma = \sigma$, $\tau\tau = 1$, or $\sigma\tau\sigma\tau\sigma\tau\sigma = \sigma\tau\sigma$.

One might think that possibly others among these fourteen operations are equal, but this is not true even for subsets of $X = \mathbb{R}$. Consider, for example, the set $Y = (0,1) \cup (1,2) \cup \{3\} \cup ((5,7) \cap \mathbb{Q})$. For each of the fourteen sets we may create a vector of 0's and 1's indicating whether the points 1, 2, 3, 4, and 6 lie in the set; the fourteen vectors are distinct (observe the conversion from binary, below), showing that the fourteen sets are distinct:

Y	$= [0,0,1,0,1]$:	5	$\tau(Y)$	$= [1,1,0,1,0]$:	26
$\sigma(Y)$	$= [1,1,1,0,1]$:	29	$\sigma\tau(Y)$	$= [1,1,1,1,1]$:	31
$\tau\sigma(Y)$	$= [0,0,0,1,0]$:	2	$\tau\sigma\tau(Y)$	$= [0,0,0,0,0]$:	0
$\sigma\tau\sigma(Y)$	$= [0,1,1,1,0]$:	14	$\sigma\tau\sigma\tau(Y)$	$= [1,1,0,0,0]$:	24
$\tau\sigma\tau\sigma(Y)$	$= [1,0,0,0,1]$:	17	$\tau\sigma\tau\sigma\tau(Y)$	$= [0,0,1,1,1]$:	7
$\sigma\tau\sigma\tau\sigma(Y)$	$= [1,1,0,0,1]$:	25	$\sigma\tau\sigma\tau\sigma\tau(Y)$	$= [0,1,1,1,1]$:	15
$\tau\sigma\tau\sigma\tau\sigma(Y)$	$= [0,0,1,1,0]$:	6	$\tau\sigma\tau\sigma\tau\sigma\tau(Y)$	$= [1,0,0,0,0]$:	16

Although $14 < 2^4$, no set of four test points suffices here, and I believe one can show that in any space X, given any subset Y and any list of four points in X, the number of distinct 4-tuples which can arise is at most 12. (This is correct: suppose A generates 14 sets and B is a subset of X that distinguishes all 14. Note, at least one of $B \cap \tau\sigma\tau(A)$, $B \cap \tau\sigma\tau(\tau(A))$ is nonempty. The strictly increasing chain

$$B \cap \tau\sigma\tau(A) \subsetneqq B \cap \tau\sigma\tau\sigma\tau\sigma\tau(A) \subsetneqq B \cap \sigma\tau\sigma\tau(A) \subsetneqq B \cap \sigma\tau\sigma\tau\sigma(A) \subsetneqq B \cap \sigma(A)$$

and its dual (with $\tau(A)$ in place of A) both hold. It follows that at least one of the inequalities $|B \cap \sigma(A)| > 4$, $|B \cap \sigma(\tau(A))| > 4$ must hold. Hence $|B| > 4$.)