Closure, Complement, and Union

11059 [2004, 64]. *Proposed by Mark Bowron, Laughlin, NV.* The [published] solution to MONTHLY problem **10577** [1998, 282] gives an example of a subset in a topological space from which infinitely many distinct sets can be obtained by repeatedly applying the three set operations of closure, complement, and union in some order. Does there exist a space containing a *finite* subset that can serve as a seed for the production of infinitely many distinct sets in this fashion?

Solution by Bruce Burdick, Roger Williams University, Bristol, RI. Put a topology on the set X of nonnegative integers by declaring that $\{0\}$ and $\{1\}$ are open and that for $n > 1$ the smallest neighborhood of n is $\{0, 1, 2, \ldots, n-2, n\}$. We claim that every singleton $\{n\}$ can be generated from $\{0\}$ with finitely many applications of the operations of closure, complement, and union.

We prove this by induction. Clearly $\{0\}$ can be so generated. Suppose for a given $n > 0$ that all singletons $\{0\}, \{1\}, \ldots, \{n-1\}$ can be generated in finitely many steps. The closure of $\{n-1\}$ is $\{n-1, n+1, n+2, \ldots\}$. The union of this with the singletons $\{0\}, \{1\}, \ldots, \{n-1\}$ is $X \setminus \{n\}$, and the complement of that is $\{n\}$.

Also solved by the proposer.